

HW #5

In all problems, f is a bounded measurable function, and E is a measurable set with $mE < \infty$.

1. Let $\alpha > 0$ and $f \geq 0$. Prove that

$$m(\{x \in E : f(x) \geq \alpha\}) \leq \frac{1}{\alpha} \int_E f$$

2. Let $f \geq 0$ and $\int_E f = 0$. Show that $f = 0$ a.e. on E .
3. Suppose that $\int_A f = 0$ for any measurable subset A of E . Show that $f = 0$ a.e. on E .
4. Suppose that $\int_a^c f = 0$ for every $c \in [a, b]$. Prove that $f = 0$ a.e. on $[a, b]$.

MATH 710, Fall 2009, HW #5, Solutions

1. Let $\alpha > 0$ and $f \geq 0$. Prove that

$$m(\{x \in E : f(x) \geq \alpha\}) \leq \frac{1}{\alpha} \int_E f$$

Solution. Let $E_\alpha = \{x \in E : f(x) \geq \alpha\}$ and $f_\alpha(x) = \alpha \chi_{E_\alpha}(x)$. Since $f \geq 0$, $f_\alpha(x) \leq f(x)$ on the set E . Therefore,

$$\alpha \cdot mE_\alpha = \int_E f_\alpha \leq \int_E f$$

and the result follows.

2. Let $f \geq 0$ and $\int_E f = 0$. Show that $f = 0$ a.e. on E .

Solution. We have

$$\{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in E : f(x) \geq \frac{1}{n}\}$$

By the previous problem (for $\alpha = \frac{1}{n}$), $m\{x \in E : f(x) \geq \frac{1}{n}\} = 0$. Therefore, by subadditivity of m , $m\{x \in E : f(x) > 0\} = 0$. The result follows, since $f \geq 0$.

3. Suppose that $\int_A f = 0$ for any measurable subset A of E . Show that $f = 0$ a.e. on E .

Solution. Let

$$A_0 = \{x \in E : f(x) = 0\}, \quad A_+ = \{x \in E : f(x) > 0\}, \quad A_- = \{x \in E : f(x) < 0\}$$

Thus, E is a disjoint union of measurable sets A_0 , A_+ , and A_- . Suppose, say, that $mA_- > 0$. Since $\int_{A_-} f = 0$ and $-f \geq 0$ on A_- , we have, by the previous problem, $-f = 0$ a.e. on A_- , that is $f = 0$ a.e. on A_- . Likewise, $f = 0$ a.e. on A_+ . The result follows, because $f = 0$ on A_0 and $E = A_0 \cup A_+ \cup A_-$.

4. Suppose that $\int_a^c f = 0$ for every $c \in [a, b]$. Prove that $f = 0$ a.e. on $[a, b]$.

Solution. Let us extend f to \mathbb{R} by setting $f(x) = 0$ for $x \notin [a, b]$. Note that $\int_I f = 0$ for any open interval I . (Prove it!) Therefore, $\int_O f = 0$ for any open set O . Since f is bounded, $|f| < M$ for some $M > 0$. Let A be a measurable subset of $[a, b]$. By Proposition 3.15(ii) (proven in class), for a given $\varepsilon > 0$, there is an open set $O \supset A$ such that $m(O \sim A) < \varepsilon/M$. Since $\int_O f = 0$, we have $\int_A f + \int_{O \sim A} f = 0$. Therefore, $|\int_A f| = |\int_{O \sim A} f| \leq \int_{O \sim A} |f| \leq M \cdot \frac{\varepsilon}{M} = \varepsilon$. It follows that $\int_A f = 0$. The result follows from the previous problem.