Decision–Making with Fuzzy Ternary Relations

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Abstract

In this paper we study three models of choice in a fuzzy environment. The first model is based on maximizing sets defined for interval scales. The second model defines a fuzzy choice set by means of a fuzzy ternary relation. Finally, we introduce an abstract model of fuzzy choice functions which extends some classical ideas of rational choice behavior. We prove that these three models generate equivalent choice mechanisms.

Keywords: Maximizing set, Fuzzy ternary relation, Fuzzy choice function.

1 Introduction

In the paper we consider the optimization problem in a general framework of best alternative choice [1]. Many problems in decision theory, especially in economical, psychological, and social applications, can be reduced to making the best choice from a set of alternatives with respect a given optimality criterion. Very often this criterion is a real–valued function on the set of all alternatives. The main objectives of the theory in question are a study of the sets of best alternatives and establishing relations between different mechanisms of choice, rather then calculating the optimum.

The classical theory of choice considers the following framework (see, for instance, [1]). Let A be a finite set of alternatives (the 'universe' of alternatives). A function $f : A \to \mathbb{R}$ is said to be a *criterion* (scale, goal function, utility function, etc.). Values of this function are assumed to be measurements in an ordinal scale [6, 7]. It means that two such functions f and g are equivalent ordinal scales if there is a strictly increasing bijection $\varphi : \mathbb{R} \to \mathbb{R}$ such that $g = \varphi \circ f$. The criterion f generates the following mechanism of choice:

$$C^f_X = \{y \in X : f(y) \geq f(x),\ \forall x \in X, \quad (1.1)$$

where $X \subseteq A$ is the set of submitted alternatives. This choice mechanism is equivalent to

$$C^f_X = \{y \in X : \text{there is no } x \in X \text{ such that } f(x) > f(y)\}. \quad (1.2)$$

For a given criterion f on A, the function $C^f : \mathbf{2}^A \to \mathbf{2}^A$ defined by (1.1) is an example of a 'choice function'. Elements of C^f_X are the 'best alternatives' in the submitted set X. According to (1.1) an alternative $x \in X$ is a best one in X if the function f assumes its maximum value on the set X at x. Thus the alternatives in C^f_X 'optimize' the criterion f. The two choice mechanisms (1.1) and (1.2) have different meanings despite their mathematical equivalence: mechanism (1.1) defines C^f_X as the set of 'dominating alternatives' as opposed to (1.2) which defines this set as the set of 'non–dominated' alternatives.

The choice mechanism defined by (1.1) can be called an 'ordinal scale' mechanism, since

$C^f_X = C^g_X$, for all $X \subset A$ if and only if f and g are equivalent ordinal scales.

One standard result in the choice theory is that a criterion (assumed to be an ordinal scale) defines a mechanism of choice which is equivalent to the pair–dominant mechanism of choice, i.e., a mechanism based on a binary relation on A. In fact, the criterion mechanism is the same as the one generated by weak ordering relations. Moreover, it is possible to describe the class of 'choice functions' in the form C^f by some characteristic properties of abstract choice functions. Here, an abstract choice function is a function in the form $C : X \mapsto Y$ where Y is a nonempty subset of X (i.e., $C_X \subseteq X$). The following is one of standard 'characteristic' properties of abstract choice functions.

Strict Heritage or Constancy condition (K):

$$X' \subseteq X,\ X' \cap C_X \neq \emptyset \ \Rightarrow\ C_{X'} = C_X \cap X'.$$

Remark. This property was introduced by Chernoff (postulate 6 in [3]. It is also known as the "weak axiom of revealed preferences" (axiom C4 in [2]). "Condition(K) requires that the options chosen from the initial set X and left in the narrowed one X' and only such options be chosen from X'..." [1].

The purpose of this paper is to suggest an extension of the classical choice theory to the case when criteria are interval scales. Let, for example, f_1 and f_2 be two criteria which are equivalent ordinal scales but there are no constants $\alpha > 0$ and β such that $f_1 = \alpha f_2 + \beta$. Then f_1 and f_2 are not equivalent interval scales, although they yield the same choice mechanisms by (1.1). Thus the classical theory is not able to distinguish optimizations in scales that are stronger than ordinal ones and we 'loose information' passing from criteria to choice functions. It turns out that there is a model of choice based on fuzzy set theory which provides the classical correspondences between the 'criterion language', the 'relation language', and the 'choice function language' for the interval scales. This model utilizes a notion of a maximizing set introduced by Zadeh in 1972 [10], (see also [4]).

We define a maximizing set for a function f on a finite set X as a fuzzy set M by its membership function $M(x)$ as follows (see [4, p.101]):

$$M(x) = \frac{f(x) - \min f}{\max f - \min f}, \qquad (1.3)$$

(we set $M = X$ if f is a constant function). Then two functions f_1 and f_2 are equivalent interval scales if and only if they define the same maximizing set by (1.3).

Intuitively, the grade of membership of $x \in X$ in M represents the degree to which f approximates to $\max f$ relative to the range of f.

Clearly, the membership function M and the function f are equivalent interval scales.

The notion of a maximizing set permits to give a proper generalization of choice mechanisms (1.1) and (1.2) that turned out to be equivalent to triple–dominant mechanisms of choice based on fuzzy ternary relations.

2 Fuzzy Choice Functions

We denote $\tilde{P}(A)$ the set of fuzzy sets with a finite universe A and $carX$ the *carrier* of a fuzzy set X, i.e., the crisp set

$$\{x \in A : X(x) > 0\}$$

where $X(x)$ is the membership function of X. We write $x \in X$ if $x \in carX$.

Definition 2.1. *A choice function C is a mapping $C : \tilde{P}(A) \to \tilde{P}(A)$ assigning a nonempty fuzzy set $C_X \subseteq X$ to every nonempty fuzzy set X with the universe A.*

The subset C_X of X is considered to be the subset of 'best' elements in X. The number $C_X(x)$ may be regarded as a 'degree of goodness' of the element $x \in X$ with respect to the choice function C.

Examples of choice functions can be found in Section 3 where a particular class of choice functions is defined by means of criteria and triple–dominant choice mechanisms. This class can be also described in an external way by some properties of abstract choice functions. One of these properties has no analog in the classical theory. This is the Separation property:

Separation (S):

$$C_X = C_{carX} \cap X, \quad \forall X \in \tilde{P}(A).$$

Intuitively, the number $C_X(x)$ is the degree to which the alternative x is 'chosen' as the 'best' one from the fuzzy set X. The Separation property asserts that this degree is determined by the choice from the crisp set $carX$ and the value $X(x)$ of the membership function of the fuzzy set X.

Another property is similar to the Strict Heritage property (K) of non fuzzy choice functions.

For a nonempty fuzzy set X, a *normalizer* of X is a fuzzy set $N(X)$ with a membership function (cf. (1.3))

$$N(X)(x) = \frac{X(x) - \min X(x)}{\max X(x) - \min X(x)},$$

if $X(x)$ is not a constant function, and $N(X) = carX$ if $X(x)$ is a constant function.

We define the Strict Heritage property (K) of a fuzzy choice function C by

Strict Heritage (K):

If $X' \subseteq X$ are nonempty crisp sets, then

$$C_{X'} = N(C_X \cap X').$$

Clearly, this property is of the same nature as the classical one described in Section 1.

3 Choice Theory for Interval Scales

In this section we present a model of optimal choice for interval scales.

Definition 3.1. *Let f be a function (criterion) on the universe of alternatives A. A* dominating subset *of a fuzzy subset $X \subseteq A$ with respect to f is a fuzzy subset D_X^f of X defined by*

$$D_X^f(x) = \frac{f(x) - m^f(X)}{M^f(X) - m^f(X)} \wedge X(x), \quad (3.1)$$

where $m^f(X) = \min_{x \in X} f$, $M^f(X) = \max_{x \in X} f$. For a constant function f, $D_X^f = X$.

Clearly, dominating subsets defined by (3.1) are invariant under positive affine transformations that define the class of interval scales [6, 7]. It is easy to prove that two criteria f_1 and f_2 are equivalent interval scales if and only if $D_X^{f_1} = D_X^{f_2}$ for all $X \in \tilde{P}(A)$. We consider D_X^f's as values of a fuzzy choice function C.

Theorem 3.1. *Let C be a fuzzy choice function. The following statements are equivalent:*

(i) *$C_X = D_X^f$ for some criterion f;*

(ii) *C satisfies properties (S) and (K).*

Proof. (i) $\Rightarrow$ (ii).

Clearly, (S) holds for C defined by (3.1). Let $X' \subseteq X$ be two crisp subsets of A. Then

$$N(D_X^f \cap X')(x) = \frac{D_X^f(x) - m^{D_X^f}(X')}{M^{D_X^f}(X') - m^{D_X^f}(X')}$$
$$= \frac{\frac{f(x)-m^f(X)}{M^f(X')-m^f(X')} - \min_{x \in X'} \frac{f(x)-m^f(X)}{M^f(X')-m^f(X')}}{\max_{x \in X'} \frac{f(x)-m^f(X)}{M^f(X')-m^f(X')} - \min_{x \in X'} \frac{f(x)-m^f(X)}{M^f(X')-m^f(X')}}$$
$$= \frac{f(x) - \min_{X'} f}{\max_{X'} f - \min_{X'} f} = D_{X'}^f(x),$$

for $x \in X'$.

Thus condition (K) holds for C.

(ii) $\Rightarrow$ (i).

Let us define $f(x) = C_A(x)$. Then

$$D_X^f(x) = \frac{f(x) - m^f(X)}{M^f(X) - m^f(X)} \wedge X(x) =$$
$$= \frac{C_A(x) - \min_{x \in X}\{C_A(x)\}}{\max_{x \in X}\{C_A(x)\} - \min_{x \in X}\{C_A(x)\}} \wedge (X(x) =$$
$$= (N(C_A \cap carX))(x) \wedge X(x) =$$
$$= C_{carX}(x) \wedge X(x) = C_X(x),$$

by properties (K) and (S). □

Thus the Separation and Strict Heritage properties completely characterize fuzzy choice functions defined by criteria which are interval scales.

It turns out that there is a mechanism of choice based on hyper dominant relations [1] which yields an equivalent description of choice in the case of interval scales. Informally, this mechanism can be described as follows. Let R be a relation between elements of A and 2–element subsets of A. Then a *weakly dominant* crisp choice function is defined by

$$C_X = \{x \in X : \text{for every } y \in X \text{ and, for at least one pair } \{y,z\} \subseteq X, \text{ holds } xR\{y,z\}\}. \quad (3.2)$$

For our purposes it is more convenient to represent R as a fuzzy ternary relation on A. Then we have the following extension of (3.2):

Definition 3.2. *Let R be a fuzzy ternary relation on A. A fuzzy choice function C_X^R based on R is defined by its membership function as follows:*

$$C_X^R(x) = \Big[\bigwedge_{y \in X} \bigvee_{z \in X} R(x,y,z)\Big] \wedge X(x) \quad (3.3)$$

In this paper we consider only those choice mechanisms (3.3) that are based on triple–dominant relations defined as follows:

Definition 3.3. *A fuzzy ternary relation R is said to be a* triple–dominant relation *if it satisfies the following conditions:*

(i) $R(x,y,z) = R(x,z,y), \quad \forall x,y,z \in A;$

(ii) $R(x,y,y) \wedge R(x,z,z) \le R(x,y,z) \le$

$\le R(x,y,y) \vee R(x,z,z);$

(iii) *There is a weak ordering $\preceq$ on A such that*

$$R(x,y,y) = R(x,x,y) = \begin{cases} 1, & \text{if } y \preceq x, \\ 0, & \text{if } x \prec y; \end{cases}$$

(iv) $R(x,u,v) = \dfrac{R(x,y,z) - R(u,y,z)}{R(v,y,z) - R(u,y,z)},$

if $y \preceq u \preceq x \le v \preceq z$ and $u \prec v$.

We define $x \prec y$ if $x \preceq y$ but $y \not\preceq x$, and $x \sim y$ if $x \preceq y$ and $y \preceq x$.

Our last theorem establishes equivalence of choice mechanisms based on interval scales and on fuzzy triple–dominant relations.

Theorem 3.2. *For a criterion f there is a fuzzy triple–dominant relation R such that $D^f = C^R$. Conversely, for any fuzzy triple–dominant relation R there is a criterion f such that $C^R = D^f$.*

Proof. (i) Let f be an interval scale on A. We define

$$R(x,y,z) = \frac{f(x) - m^f(\{x,y,z\})}{M^f(\{x,y,z\}) - m^f(\{x,y,z\})}, \quad (3.4)$$

if f is not a constant function on $\{x,y,z\}$, and

$$R(x,y,z) = 1, \quad (3.5)$$

if f is a constant function on $\{x,y,z\}$.

Thus defined R is a triple–dominant relation. Indeed, property (i) in Definition 3.3 clearly holds. We have

$$R(x,y,y) = \begin{cases} 1, & \text{if } f(x) \ge f(y), \\ 0, & \text{if } f(x) < f(y), \end{cases} \quad (3.6)$$

and the same equation holds for $R(x,x,y)$. It is easy to see now that property (ii) in Definition 3.3 holds.

Let us define a relation $\preceq$ on A by

$$x \le y \iff f(x) \le f(y).$$

Thus defined $\preceq$ is a weak ordering on A. Clearly, property (iii) in Definition 3.3 holds.

Finally, let us verify property (iv) in Definition 3.3. Suppose that $y \preceq u \preceq x \preceq v \preceq z$ and $u \prec v$. Then

$$f(y) \le f(u) \le f(x) \le f(v) \le f(z)$$

and $f(u) < f(v)$ and we have

$$R(v,y,z) = \frac{f(v) - f(y)}{f(z) - f(y)}$$
$$R(u,y,z) = \frac{f(u) - f(y)}{f(z) - f(y)}$$
$$R(x,y,z) = \frac{f(x) - f(y)}{f(z) - f(y)}$$
$$R(x,u,v) = \frac{f(x) - f(u)}{f(v) - f(u)}$$

Hence,

$$\frac{R(x,y,z)-R(u,y,z)}{R(v,y,z)-R(u,y,z)} = $$
$$= \frac{\frac{f(x)-f(y)}{f(z)-f(y)} - \frac{f(u)-f(y)}{f(z)-f(y)}}{\frac{f(v)-f(y)}{f(z)-f(y)} - \frac{f(u)-f(y)}{f(z)-f(y)}} =$$
$$= \frac{f(x)-f(u)}{f(v)-f(u)} = R(x,u,v).$$

It remains to show that $C^R = D^f$.

For given $x, y \in X$ we have

$$\bigvee_{z \in X} R(x,y,z) = 1,$$

if $f(y) \leq f(x)$. Indeed, for $z = y$ we have, by (3.6), $R(x,y,z) = 1$.

Suppose now that $f(y) > f(x)$. If $f(z) \geq f(x)$, then $R(x,y,z) = 0$. If $f(z) < f(x)$, then

$$R(x,y,z) = \frac{f(x)-f(z)}{f(y)-f(z)}.$$

Therefore,

$$\bigvee_{z \in X} R(x,y,z) = \bigvee_{z \in X} \frac{f(x)-f(z)}{f(y)-f(z)} =$$
$$= \frac{f(x)-f(z_0)}{f(y)-f(z_0)},$$

where $f(z_0) = m^f(X)$.

Now, for $f(y_0) = M^f(X)$, we have

$$C^R_X(x) = \Big[\bigwedge_{y \in X} \bigvee_{z \in X} R(x,y,z)\Big] \wedge X(x) =$$
$$= \frac{f(x)-f(z_0)}{f(y_0)-f(z_0)} \wedge X(x) = D^f_X(x).$$

(ii) Suppose now that R is a triple–dominant relation on A. We define a function f on A by

$$f(x) = R(x, y_0, z_0), \tag{3.7}$$

where y_0 and z_0 are maximal and minimal elements in A with respect to the relation $\preceq$. By part (i) of the proof, it suffices to prove (3.4) and (3.5) for the function f defined by (3.7).

First we prove that (3.7) defines an increasing function with respect to $\preceq$. More precisely, we prove that

$$x \prec y \quad \Leftrightarrow \quad f(x) < f(y)$$

and

$$x \sim y \quad \Leftrightarrow \quad f(x) = f(y).$$

Suppose $x \preceq y$. Then $z_0 \preceq x \preceq y \preceq y_0$. If $z_0 \prec y$, then, by conditions (iv), (iii), and (i) of Definition 3.3,

$$R(x, z_0, y) = \frac{R(x,z_0,y_0) - R(z_0,z_0,y_0)}{R(y,z_0,y_0) - R(z_0,z_0,y_0)} =$$
$$= \frac{R(x,z_0,y_0)}{R(y,z_0,y_0)} = \frac{f(x)}{f(y)} \leq 1.$$

Hence, $f(x) \leq f(y)$.

If $x \prec y_0$, then, by conditions (iv), (i) and (iii) of Definition 3.3,

$$R(y, x, y_0) = \tag{3.8}$$
$$= \frac{R(y,z_0,y_0) - R(x,z_0,y_0)}{R(y_0,z_0,y_0) - R(x,z_0,y_0)} =$$
$$= \frac{f(y)-f(x)}{1-f(x)} \geq 0,$$

Thus $f(x) \leq f(y)$, since condition (iv) of Definition 3.3 implies

$$R(x, z_0, y_0) \neq R(y_0, z_0, y_0).$$

Suppose now that $y \preceq z_0$ and $y_0 \preceq x$. Then $z_0 \sim x \sim y \sim y_0$ and by conditions (ii) and (iii) of Definition 3.3,

$$R(x, y_0, z_0) \geq R(x, y_0, y_0) \wedge R(x, z_0, z_0) = 1$$

and

$$R(y, y_0, z_0) \geq R(y, y_0, y_0) \wedge R(y, z_0, z_0) = 1,$$

i.e., $f(x) = f(y) = 1$.

We proved that f is an increasing function on A with respect to the relation $\preceq$.

Finally, let $x \prec y$. Then, by condition (iv) of Definition 3.3 applied to $z_0 \preceq x \preceq x \prec y \preceq y_0$,

$$R(x, y_0, z_0) \neq R(y, y_0, z_0).$$

Hence, $f(x) < f(y)$.

Suppose f is a constant function on $\{x, y, z\}$. Then $x \sim y \sim z$ and, by conditions (ii) and (iii) of Definition 3.3, $R(x,y,z) = 1$, i.e., (3.5) holds.

Now we prove that (3.4) holds for f defined by (3.7) and R. Both sides of (3.4) are symmetric with respect to y and z. Hence it suffices to consider the case when $y \preceq z$. We consider the following possible cases.

$y \sim z$. Then the right side of (3.4) is 1 if $y \sim z \prec x$, and is 0 if $x \prec y \sim z$. By conditions (iii) and (iv) of Definition 3.3, the left side of (3.4) takes the same values.

$x \prec y \prec z$. Then $R(x, y, z) = 0$, by conditions (iii) and (iv) of Definition 3.3; the right side of (3.4) equals zero because $f(x)$ is the minimal value of f on $\{x, y, z\}$.

$y \preceq x \preceq z$ and $y \prec z$. Then, by condition (iv) of Definition 3.3,

$$R(x,y,z) = \frac{R(x,z_0,y_0) - R(y,z_0,y_0)}{R(z,z_0,y_0) - R(y,z_0,y_0)} = \\ = \frac{f(x) - f(y)}{f(z) - f(y)} = \\ = \frac{f(x) - m^f(\{x,y,z\})}{M^f(\{x,y,z\}) - m^f(\{x,y,z\})}$$

$y \prec z \preceq x$. By conditions (iii) and (iv) of Definition 3.3, $R(x, y, z) = 1$; the right side of (3.4) equals 1, since $f(x)$ is the maximal value of f on $\{x, y, z\}$.

□

4 Conclusion

We have shown (Theorem 3.2) that choice mechanisms based on criteria which are measurements in interval scales can be equivalently described as choice mechanisms based on fuzzy triple–dominant relations.

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