Metric Geometry of Hypercubes

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Abstract

We show that a hypercube is a homogeneous metric space with respect to the class of partial subcubes.

1 Introduction

In [5] Djoković characterized connected (finite or infinite) graphs that can be isometrically embedded into a hypercube. These graphs are known as *partial cubes* [10].

Hypercubes and partial cubes enjoy reach metric structures. In this paper we use general principles of classical distance geometry to investigate homogeneity properties of hypercubes. For a general geometric background of this approach the reader is referred to [3].

Let X and Y be metric spaces. An *isometry* from X onto Y is a distance preserving bijection $X \hookrightarrow Y$. A metric space X is *homogeneous* if for any two points $x, y \in X$ there is an isometry $\alpha : X \hookrightarrow X$ such that $\alpha(x) = y$. Following [4], we say that a metric space X is *fully homogeneous* if, for every two metric subspaces $A, B \subseteq X$ and an isometry $A \hookrightarrow B$, this isometry can be extended to an isometry of the entire space X onto itself. Euclidean, spherical, and hyperbolic spaces are examples of fully homogeneous metric spaces [1]. Another example is a finite metric space X with the discrete metric. Note that such a space is not fully homogeneous if X is an infinite set.

In a more general setting (cf. [3]), let $\mathfrak{K}$ be a nonempty class of metric subspaces of a given metric space X. We say that the space X is *homogeneous with respect to* $\mathfrak{K}$ (or $\mathfrak{K}$*–homogeneous*) if, for every two subspaces $A, B \in \mathfrak{K}$ and an isometry $A \hookrightarrow B$, this isometry can be extended to an isometry of the entire space X onto itself. For example, a finite dimensional normed vector space is homogeneous with respect to the family of its vector subspaces (Witt's Theorem [11]). It is shown in [12] that some ℓ_1–spaces are homogeneous with respect to classes of rectangular subspaces.

We begin by introducing basic facts about hypercubes and partial cubes in the next section.

Vertices of partial cubes form well graded families of sets. On the other hand, well graded families of subsets of a set X induce partial cubes on X. This relationship is crucial for our investigation of homogeneity properties of hypercubes. In Section 3 we introduce some geometric properties of well graded families of sets.

The isometry group of a hypercube is characterized in Section 4. For reader's convenience we prove all statements, although, for finite dimensional hypercubes, some of these statements are found elsewhere in the literature.

Finally, in Section 5 we show that a hypercube is a homogeneous metric space with respect to a class of its partial subcubes.

2 Hypercubes and partial cubes

Let X be a set. We denote $\mathcal{P}_f(X)$ the set of all finite subsets of X. A graph $\mathcal{H}(X)$ has the set $\mathcal{P}_f(X)$ as the set of its vertices; a pair of vertices $\{P, Q\}$ is an edge of $\mathcal{H}(X)$ if the symmetric difference $P \Delta Q$ is a singleton. The graph $\mathcal{H}(X)$ is called the *hypercube on* X [5].

Theorem 2.1. *Two hypercubes $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are isomorphic if and only if the sets X and Y have the same cardinality.*

Proof. (Necessity.) Let $\alpha : \mathcal{P}_f(X) \to \mathcal{P}_f(Y)$ be a bijection defining an isomorphism of $\mathcal{H}(X)$ onto $\mathcal{H}(Y)$. Since $\{\varnothing, \{x\}\}$ is an edge of $\mathcal{H}(X)$ for $x \in X$, there is a unique $y \in Y$ such that either $\alpha(\{x\}) = \alpha(\varnothing) \cup \{y\}$ for $y \notin \alpha(\varnothing)$, or $\alpha(\{x\}) = \alpha(\varnothing) \setminus \{y\}$ for $y \in \alpha(\varnothing)$. Clearly, the function $x \mapsto y$ is an injection from X into Y. Thus, $|X| \leq |Y|$. By symmetry, $|X| = |Y|$.

(Sufficiency.) Let β be a bijection from X onto Y and let $\alpha : \mathcal{P}_f(X) \to \mathcal{P}_f(Y)$ be defined by

$$\alpha(P) = \beta(P) \quad \text{for } P \in \mathcal{P}_f(X).$$

It is clear that α defines an isomorphism between $\mathcal{H}(X)$ and $\mathcal{H}(Y)$. □

If X is a finite set of cardinality n, then the graph $\mathcal{H}(X)$ is the *n–cube* Q_n.

The shortest path distance $d(P, Q)$ on the hypercube $\mathcal{H}(X)$ is the *Hamming distance* between sets P and Q:

$$d(P, Q) = |P \Delta Q| \quad \text{for } P, Q \in \mathcal{P}_f.$$

The set $\mathcal{P}_f(X)$ is a metric space with the metric d.

Let $\mathcal{F} \subseteq \mathcal{P}_f(X)$ be a nonempty family of finite subsets of X and let G be a subgraph of $\mathcal{H}(X)$ induced by $\mathcal{F}$. If G is an isometric subgraph of $\mathcal{H}(X)$, that is, the shortest path distance d_G on G is the Hamming distance, then G is called a *partial cube on* X [10]. In general, a graph is a *partial cube* if it can be isometrically embedded into $\mathcal{H}(X)$ for some set X.

Let $G = (V, E)$ be a (simple) connected graph. For an edge $\{u, v\} \in E$, let W_{uv} be a set of vertices of G that are closer to u than to v:

$$W_{uv} = \{w \in V : d_G(w, u) < d_G(w, v)\}.$$

If G is bipartite, then sets W_{uv} and W_{vu} partition V. Following Eppstein [8], we call these sets and graphs induced by these sets *semicubes* of G.

The following theorem is due to Djoković [5].

Theorem 2.2. *A connected graph G is a partial cube if and only if G is bipartite and all its semicubes are convex subgraphs of G.*

Let $G = (V, E)$ be a connected bipartite graph. Two edges of G are in relation θ (*Djoković's relation*) if their respective semicubes define the same partition of V. If G is a partial cube, then θ is an equivalence relation on E [5].

The (isometric) *dimension*, $\dim G$, of a graph G is the smallest cardinality of a set X such that G can be isometrically embedded into the hypercube $\mathcal{H}(X)$. The following theorem is Theorem 2 in [5].

Theorem 2.3. *Let G be a partial cube. Then $\dim G = |E/\theta|$, where E/θ is the set of equivalence classes of θ.*

3 Partial cubes and well graded families of sets

By definition, a family $\mathcal{F}$ of finite subsets of a set X induces a partial cube on X if for any two distinct $P, Q \in \mathcal{F}$ there is a sequence of sets $R_0 = P, R_1, \ldots, R_n = Q$ in $\mathcal{F}$ such that

$$d(R_i, R_{i+1}) = 1 \quad \text{for all } 0 \leq i < n, \text{ and} \quad d(P, Q) = n. \tag{3.1}$$

The families of sets satisfying these conditions are known as *well graded families* of sets [7]. We shall call them *wg–families* of sets. Note that the above–mentioned sequence (R_i) is a shortest path from P to Q in $\mathcal{H}(X)$ (and in the subgraph induced by $\mathcal{F}$).

In the rest of the paper we use the same symbol, say $\mathcal{F}$, for a family of finite subsets of X and a subgraph of $\mathcal{H}(X)$ induced by this family. We consider $\mathcal{F}$ as a metric space with the Hamming distance as its metric. To avoid trivialities we assume that $|\mathcal{F}| \geq 2$.

We say that a set $R \in \mathcal{P}_f(X)$ is *lattice between* sets $P, Q \in \mathcal{P}_f(X)$ if

$$P \cap Q \subseteq R \subseteq P \cup Q,$$

and we say that it is *metrically between* P and Q if

$$d(P, R) + d(R, Q) = d(P, Q).$$

It is known (see, for instance, [2]) that the lattice and metric betweenness relations coincide on $\mathcal{P}_f(X)$, so we can simply speak about the *betweenness relation* on $\mathcal{F} \subseteq \mathcal{P}_f(X)$.

Let $\mathcal{F}$ be a nonempty family of sets in $\mathcal{P}_f(X)$. The set of all $R \in \mathcal{F}$ that lie between $P, Q \in \mathcal{F}$ is the *interval* $\mathcal{I}(P, Q)$ *between* P *and* Q *in* $\mathcal{F}$. Note that the set $\mathcal{I}(P, Q)$ is defined relative to the family $\mathcal{F}$. In general, the interval between P and Q in $\mathcal{F}$ is a subset of the interval between P and Q in $\mathcal{P}_f(X)$.

Two distinct sets $P, Q \in \mathcal{F}$ are *adjacent in* $\mathcal{F}$ if $\mathcal{J}(P, Q) = \{P, Q\}$. If sets P and Q form an edge in the graph induced by $\mathcal{F}$, then P and Q are adjacent in $\mathcal{F}$ but, generally speaking, not vice versa. The following theorem gives a 'local' characterization of wg–families of sets.

Theorem 3.1. *A family* $\mathcal{F} \subseteq \mathcal{P}_f(X)$ *is well graded if and only if* $d(P, Q) = 1$ *for any two sets* P *and* Q *that are adjacent in* $\mathcal{F}$.

Proof. (Necessity.) Let $\mathcal{F}$ be a wg–family of sets. Suppose that P and Q are adjacent in $\mathcal{F}$. There is a sequence $R_0 = P, R_1, \ldots, R_n = Q$ that satisfies conditions (3.1). Since the sequence (R_i) is a shortest path in $\mathcal{F}$, we have

$$d(P, P_i) + d(P_i, Q) = d(P, Q) \quad \text{for all } 0 \leq i \leq n.$$

Thus, $P_i \in \mathcal{I}(P, Q) = \{P, Q\}$. It follows that $d(P, Q) = n = 1$.

(Sufficiency.) Let P and Q be two distinct sets in $\mathcal{F}$. We prove by induction on $n = d(P, Q)$ that there is a sequence $(R_i) \in \mathcal{F}$ satisfying conditions (3.1).

The statement is trivial for $n = 1$. Suppose that $n > 1$ and that the statement is true for all $k < n$. Let P and Q be two sets in $\mathcal{F}$ such that $d(P, Q) = n$. Since $d(P, Q) > 1$, the sets P and Q are not adjacent in $\mathcal{F}$. Therefore there exists $R \in \mathcal{F}$ that lies between P and Q and is distinct from these two sets. Then $d(P, R) + d(R, Q) = d(P, Q)$ and both distances $d(P, R)$ and $d(R, Q)$ are less than n. By the induction hypothesis, there is a sequence $(R_i) \in \mathcal{F}$ such that

$$P = R_0, \ R = R_j, \ Q = R_n \qquad \text{for some } 0 < j < n,$$

satisfying conditions (3.1) for $0 \leq i < j$ and $j \leq i < n$. It follows that $\mathcal{F}$ is a wg–family of sets. $\square$

Let G be a graph which is a partial cube, that is, it can be isometrically embedded into a hypercube. This graph admits isometric representations as partial cubes on various sets X. For instance, the graph K_2 can be isometrically embedded in different ways into any hypercube $\mathcal{H}(X)$ with $|X| > 1$. It is desirable to 'minimize' the class of hypercubes $\mathcal{H}(X)$ that can be used as target graphs for isometric embeddings of G. We do it by 'reducing the domain' of a wg–family of sets.

Let $\mathcal{F}'$ be a family of subsets of a set X'. We define the *reduction* of $\mathcal{F}'$ as a family $\mathcal{F}$ of subsets of $X = \cup\mathcal{F}' \setminus \cap\mathcal{F}'$ consisting of the intersections of sets in $\mathcal{F}'$ with X. It is clear that $\mathcal{F}$ satisfies the following two conditions

$$\cap\mathcal{F} = \varnothing \quad \text{and} \quad \cup\mathcal{F} = X. \tag{3.2}$$

In other words, the reduction $\mathcal{F}$ of $\mathcal{F}'$ is obtained by eliminating 'inactive' elements from the set X'.

Proposition 3.1. *The partial cubes induced by a wg–family $\mathcal{F}'$ and its reduction $\mathcal{F}$ are isomorphic.*

Proof. It suffices to prove that metric spaces $\mathcal{F}'$ and $\mathcal{F}$ are isometric. Let us define a mapping $\alpha : \mathcal{F}' \to \mathcal{F}$ by $P \mapsto P \cap X$. Clearly, α is surjective. We have

$$(P \cap X)\Delta(Q \cap X) = (P\Delta Q) \cap X = (P\Delta Q) \cap (\cup\mathcal{F}' \setminus \cap\mathcal{F}') = P\Delta Q.$$

Thus, $d(\alpha(P), \alpha(Q)) = d(P, Q)$. Consequently, α is an isometry. ☐

Let G be a partial cube on some set X induced by a wg–family $\mathcal{F}$ and let $\{P, Q\}$ be an edge of G. Then there is $x \in \cup\mathcal{F} \setminus \cap\mathcal{F}$ such that $P\Delta Q = \{x\}$. It is not difficult to show that two sets

$$\{R \in \mathcal{F} : x \in R\} \quad \text{and} \quad \{R \in \mathcal{F} : x \notin R\}$$

form the same partition of $\mathcal{F}$ as semicubes W_{PQ} and W_{QP}. Thus there is one–to–one correspondence between equivalence classes of θ and those elements $x \in X$ that define edges in G.

Lemma 3.1. *If $\mathcal{F}$ is a wg–family of sets, then for any $x \in \cup\mathcal{F} \setminus \cap\mathcal{F}$ there are sets $P, Q \in \mathcal{F}$ such that $P\Delta Q = \{x\}$.*

Proof. For a given $x \in \cup\mathcal{F} \setminus \cap\mathcal{F}$ there are sets S and T in $\mathcal{F}$ such that $x \in S$ and $x \notin T$. Let $R_0 = S, R_1, \dots, R_n = T$ be a sequence of sets in $\mathcal{F}$ satisfying conditions (3.1). It is clear that there is i such that $x \in R_i$ and $x \notin R_{i+1}$. Hence, $R_i\Delta R_{i+1} = \{x\}$, so we can choose $P = R_i$ and $Q = R_{i+1}$. ☐

It follows that any element of the set $\cup\mathcal{F} \setminus \cap\mathcal{F}$ defines an edge of G. By applying Theorem 2.3, we have the following result.

Theorem 3.2. *Let $\mathcal{F}$ be a wg–family of finite subsets of a set X and G be the partial cube induced by $\mathcal{F}$. Then*

$$\dim G = |\cup\mathcal{F} \setminus \cap\mathcal{F}|.$$

By Proposition 3.1, we may assume that the wg–family $\mathcal{F}$ satisfies conditions (3.2). Then we can reformulate Theorem 3.2 as follows:

Theorem 3.3. *Let $\mathcal{F}$ be a wg–family of finite subsets of a set X satisfying conditions (3.2) and let G be the partial cube induced by $\mathcal{F}$. Then*

$$\dim G = |X|.$$

The result of Theorem 3.2 suggests the definition of the dimension of an arbitrary family $\mathcal{F}$ of subsets of a given set X as the cardinality of the set $\cup\mathcal{F} \setminus \cap\mathcal{F}$:

$$\dim \mathcal{F} = |\cup\mathcal{F} \setminus \cap\mathcal{F}|.$$

4 The isometry group of a hypercube

Let X be a set and let $Iso(\mathcal{H}(X))$ be the *isometry group* of the hypercube $\mathcal{H}(X)$, that is, the group of all isometries of the metric space $\mathcal{P}_f(X)$ onto itself.

For $P \in \mathcal{P}_f(X)$ we define a function α_P from $\mathcal{P}_f(X)$ onto itself by

$$\alpha_P(S) = S\Delta P \quad \text{for } S \in \mathcal{P}_f(X).$$

We have

$$d(\alpha_P(S), \alpha_P(T)) = |S\Delta P\Delta T\Delta P| = |S\Delta T| = d(S,T).$$

Thus α_P is an isometry of $\mathcal{P}_f$ onto itself. Clearly, the isometries α_P form a subgroup K of $Iso(\mathcal{H}(X))$ with the identity element $e = \alpha_\varnothing$.

Let π be a permutation on X, that is, a bijection from X onto itself. The permutation π defines an isometry $\hat{\pi} : \mathcal{P}_f(X) \hookrightarrow \mathcal{P}_F(X)$ by

$$\hat{\pi}(S) = \pi(S) = \{\pi(x) : x \in S\} \quad \text{for } S \in \mathcal{P}_f(X).$$

These isometries form a subgroup H of $Iso(\mathcal{H}(X))$.

Theorem 4.1. *The isometry group of the hypercube $\mathcal{H}(X)$ is a semidirect product of the subgroup K by the subgroup H:*

$$\operatorname{Iso}(\mathcal{H}(X)) = K \rtimes H.$$

Proof. First we prove that $Iso(\mathcal{H}(X))$ is generated by subgroups K and H.

Let $\alpha : \mathcal{H}(X) \hookrightarrow \mathcal{H}(X)$ be an isometry and let $P = \alpha^{-1}(\varnothing)$. Clearly, $\alpha_P \circ \alpha^{-1}(\varnothing) = \varnothing$. Since $d(\varnothing, \{x\}) = 1$ for any $x \in X$, the isometry $\alpha_P \circ \alpha^{-1}$ defines a permutation π on X such that $\alpha_P \circ \alpha^{-1}(\{x\}) = \{\pi(x)\}$. Let us define $\beta = \hat{\pi}^{-1} \circ \alpha_P \circ \alpha^{-1}$. Then $\beta(\{x\}) = \{x\}$ for all $x \in X$. For $S \in \mathcal{P}_f(X)$ we have

$$|\beta(S)| = d(\beta(S), \varnothing) = d(S, \varnothing) = |S|,$$

since $\beta(\varnothing) = \varnothing$. For $x \in \beta(S)$ we have

$$d(\{x\}, S) = d(\{x\}, \beta(S)) = |\beta(S)| - 1 = |S| - 1,$$

which is possible only if $x \in S$. Therefore, $\beta(S) \subseteq S$. The same argument shows that $S \subseteq \beta(S)$. Thus β is the identity. It follows that $\alpha = \hat{\pi}^{-1} \circ \alpha_P$, that is, $Iso(\mathcal{H}(X)) = KH$.

Clearly, $K \cap H = \{e\}$. To prove that $Iso(\mathcal{H}(X)) = K \rtimes H$ we need to show that K is a normal subgroup of $Iso(\mathcal{H}(X))$ [9, Theorem 6.5.3].

We have for $\alpha_P \in K$ and $\hat{\pi} \in H$

$$\hat{\pi} \circ \alpha_P \circ \hat{\pi}^{-1}(S) = \pi(\pi^{-1}(S)\Delta P) = S\Delta\pi(P) = \alpha_{\pi(P)}(S) \quad \text{for } S \in \mathcal{P}_f(X).$$

Thus $\hat{\pi} \circ \alpha_P \circ \hat{\pi}^{-1} = \alpha_{\pi(P)}$, which means that K is a normal subgroup. □

It is clear that the group H is isomorphic to the symmetric group $S(X)$, that is, the group of all permutations on X.

The set $\mathcal{P}_f(X)$ is a commutative group under the operation of symmetric difference of sets. The empty set is the identity of this group. Obviously, the group K is isomorphic to the group $\mathcal{P}_f(X)$. Since $P\Delta P = \varnothing$, all elements of the group $\mathcal{P}_f(X)$ have order 2. Thus $\mathcal{P}_f(X)$ is an elementary Abelian two–group.

Let $P = \{x_1, \dots, x_n\}$. Then

$$P = \{x_1\}\Delta \cdots \Delta\{x_n\}.$$

Hence, $\alpha_P = \alpha_{\{x_1\}} \circ \cdots \circ \alpha_{\{x_n\}}$. It follows that the family $\{\alpha_{\{x\}}\}_{x\in X}$ is a set of generators of $\mathcal{P}_f(X)$. These generators satisfy relations $\alpha^2_{\{x\}} = e$ for all $x \in X$.

The properties of the group $\mathcal{P}_f(X)$ established in the preceding paragraph characterize this group.

Theorem 4.2. *Let G be a group generated by a family of elements $\{g_x\}_{x\in X}$ satisfying the relations (and only these relations):*

(i) $g_x g_y = g_y g_x$,

(ii) $g_x^2 = e$

for all $x, y \in X$. Then G is isomorphic to $\mathcal{P}_f(X)$.

Proof. By relations (i) and (ii), any element $g \neq e$ of G can be written in the form

$$g = g_{x_1} \cdots g_{x_n},$$

where all elements $x_1, \dots, x_n \in X$ are distinct. Moreover, this representation is unique up to permutations of these elements. We define a mapping φ from G to $\mathcal{P}_f(X)$ by

$$\varphi(g) = \{x_1, \dots, x_n\} \quad \text{and} \quad \varphi(e) = \varnothing$$

for $g = g_{x_1} \cdots g_{x_n}$. It is clear that φ is a bijection.

Let $g = g_{x_1} \cdots g_{x_n}$ and $h = g_{y_1} \cdots g_{y_k}$. Then

$$gh = g_{x_1} \cdots g_{x_n} g_{y_1} \cdots g_{y_k} = g_{z_1} \cdots g_{z_m}.$$

By relations (i) and (ii), $\{z_1, \dots, z_m\} = \{x_1, \dots, x_n\}\Delta\{y_1, \dots, y_k\}$. Thus, φ is an isomorphism from G onto $\mathcal{P}_f(X)$. □

Remark 4.1. Let $X = \{a_1, \dots, a_n\}$, so $\mathcal{H}(X) = Q_n$. Let us identify the hypercube Q_n with 0/1–vectors in the space $\mathbb{R}^n$ endowed with the ℓ_1 metric. It is clear that isometries $\alpha_{\{a_i\}}$ are restrictions to Q_n of reflections in hyperplanes $x_i = 1/2$. Thus, the isometry group $Iso(Q_n)$ is generated by these reflections and permutations of axes in the space $\mathbb{R}^n$. In particular, the order of the isometry group of the n–cube is $2^n n!$. These are well known facts in the theory of finite dimensional hypercubes.

It is interesting to note that the vertices of Q_n form an n–dimensional vector space V over the field F_2. The transformations α_P form the translation group K of V and elements of the group H can be regarded as 'orthogonal' transformations of V. Then Theorem 4.1 is a finite analog of a classical result in geometry: The group of motions of the n–dimensional Euclidean space is a semidirect product of its translation group by the orthogonal group $O(n)$.

5 Homogeneity properties of hypercubes

We begin with an example demonstrating that, generally speaking, a metric space $\mathcal{H}(X)$ is not fully homogeneous.

Example 5.1. Let $X = \{a, b, c, d\}$. Consider two families of subsets of X:

$$\mathcal{A} = \{\varnothing, \{a, d\}, \{b, d\}, \{c, d\}\}$$

and

$$\mathcal{B} = \{\varnothing, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

Clearly, $\mathcal{A}$ and $\mathcal{B}$ are isometric. The distance from the set $\{d\}$ to all sets in $\mathcal{A}$ is 1. On the other hand, it is easy to verify that there is no subset of X which is on distance 1 from all sets in $\mathcal{B}$ (see Figure 5.1). Thus an isometry from $\mathcal{A}$ onto $\mathcal{B}$ cannot be extended to an isometry from $\mathcal{H}(X)$ onto itself. Note that $\dim \mathcal{A} = 4$ but $\dim \mathcal{B} = 3$.

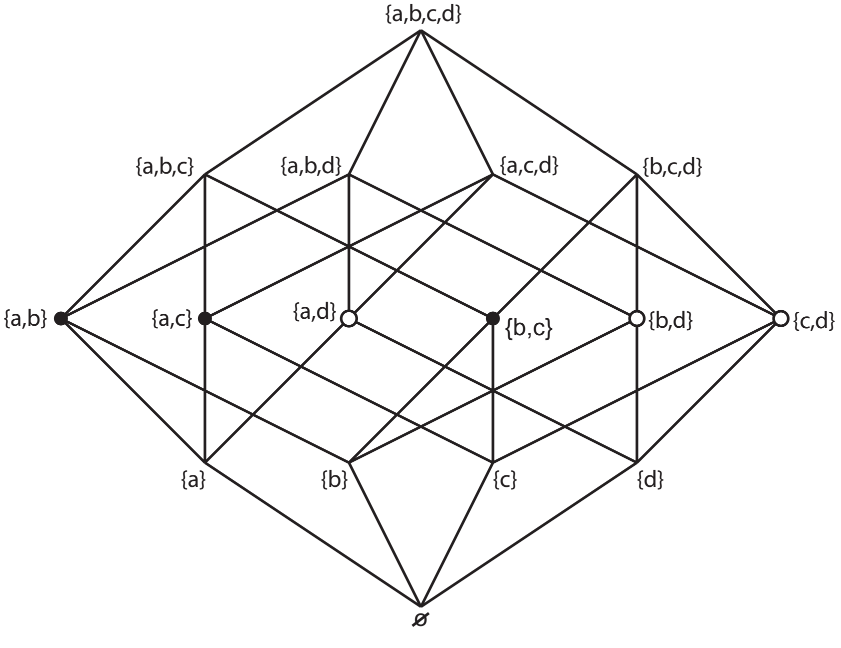

Figure 5.1: Families $\mathcal{A}$ and $\mathcal{B}$ in the hypercube Q_4.

The above argument can be easily modified to show that $\mathcal{H}(X)$ is not fully homogeneous if $|X| > 3$. It can be shown that $\mathcal{H}(X)$ is fully homogeneous if

$|X| \leq 3$ (we omit the proof). Note that any metric space $\mathcal{H}(X)$ is homogeneous. Indeed, for $P, Q \in \mathcal{P}_f(X)$ we have $Q = \alpha_Q \circ \alpha_P(P)$.

The next theorem is a consequence of Theorem 19.1.2 in [6]. It is also a special case of Theorem 5.2.

Theorem 5.1. *Let X be a finite set and $\mathfrak{K}$ be the class of all partial cubes on X of dimension $|X|$. The metric space $\mathcal{H}(X)$ is $\mathfrak{K}$–homogeneous.*

Example 5.2. Let Y be a proper subset of an infinite set X such that $|Y| = |X|$. By Theorem 2.1, the hypercubes $\mathcal{H}(Y)$ and $\mathcal{H}(X)$ are isomorphic. These hypercubes are partial cubes on X of the same dimension $|X|$. It is clear that an isometry $\mathcal{H}(Y) \hookrightarrow \mathcal{H}(X)$ cannot be extended to an isometry of $\mathcal{H}(X)$ onto itself. Thus Theorem 5.1 does not hold for infinite sets X.

It is convenient to formulate the main result of this section in terms of wg–families of finite subsets of a given set X.

Let $\mathcal{K}$ be the family of all nonempty subsets of X satisfying the following condition:

$$|Y| = |Z| \quad \Rightarrow \quad |X \setminus Y| = |X \setminus Z| \quad \text{for all } Y, Z \in \mathcal{K}. \tag{5.1}$$

Note that $\mathcal{K}$ contains all finite subsets of X and the set X itself.

For $\mathcal{F} \subseteq \mathcal{P}_f(X)$ we denote $D(\mathcal{F}) = \cup \mathcal{F}$, the *domain* of $\mathcal{F}$, and define

$$\mathfrak{K} = \{\mathcal{F} \subseteq \mathcal{P}_f(X) : \mathcal{F} \text{ is well graded and } D(\mathcal{F}) \in \mathcal{K}\}. \tag{5.2}$$

Theorem 5.2. *The metric space $\mathcal{P}_f(X)$ is $\mathfrak{K}$–homogeneous.*

We obtain the result of this theorem after proving few lemmas.

A general remark is in order. Let Y be a homogeneous metric space, A and B be two subspaces of Y, and α be an isometry from A onto B. Let c be a fixed point in Y. For a given $a \in A$, let $b = \alpha(a) \in B$. Since Y is homogeneous, there are isometries β and γ from Y onto itself such that $\beta(a) = c$ and $\gamma(b) = c$. Then $\lambda = \gamma\alpha\beta^{-1}$ is an isometry from $\beta(A)$ onto $\gamma(B)$ such that $\lambda(c) = c$. Clearly, α is extendable to an isometry of Y if and only if λ is extendable.

Thus, in the case of the space $\mathcal{P}_f(X)$, we may consider only wg–families of subsets containing the empty set $\varnothing$ and isometries between these families fixing this point.

For a given family $\mathcal{F} \subseteq \mathcal{P}_f(X)$ we define a function $r_{\mathcal{F}} : D(\mathcal{F}) \to \mathbb{N}$ by

$$r_{\mathcal{F}}(x) = \min\{|R| : x \in R,\ R \in \mathcal{F}\}.$$

For $k \in \mathbb{N}$ a subset $X_k^{\mathcal{F}}$ of X is defined by

$$X_k^{\mathcal{F}} = \{x \in X : r_{\mathcal{F}}(x) = k\}.$$

We have $X_i^{\mathcal{F}} \cap X_j^{\mathcal{F}} = \varnothing$ for $i \neq j$, and $\cup_k X_k^{\mathcal{F}} = D(\mathcal{F})$. Note that some of the sets $X_k^{\mathcal{F}}$ could be empty for $k > 1$.

Example 5.3. Let $X = \{a, b, c\}$ and $\mathcal{F} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. We have $r_{\mathcal{F}}(a) = r_{\mathcal{F}}(b) = 1$, $r_{\mathcal{F}}(c) = 3$ and

$$X_1^{\mathcal{F}} = \{a, b\},\ X_2^{\mathcal{F}} = \varnothing,\ X_3^{\mathcal{F}} = \{c\}.$$

Lemma 5.1. *The set $X_1^{\mathcal{F}}$ is not empty and for any nonempty set $P \in \mathcal{F}$ there is $x \in P$ such that $P \setminus \{x\} \in \mathcal{F}$.*

Proof. Since $\mathcal{F}$ is well graded and contains the empty set, there is a nested sequence $\varnothing, R_1, \ldots, R_k = P$ of distinct sets in $\mathcal{F}$ such that $|R_{i+1} \setminus R_i| = 1$. Since R_1 is a singleton, we have $X_1^{\mathcal{F}} \neq \varnothing$. Clearly, $R_{k-1} = P \setminus \{x\}$ for some $x \in P$. Thus $P \setminus \{x\} \in \mathcal{F}$. $\square$

Lemma 5.2. *For $P \in \mathcal{F}$ and $x \in P$ we have*

$$r_{\mathcal{F}}(x) = |P| \quad \Rightarrow \quad P \setminus \{x\} \in \mathcal{F}.$$

Proof. By Lemma 5.1, there is $y \in P$ such that $P \setminus \{y\} \in \mathcal{F}$. Since

$$|P \setminus \{y\}| = |P| - 1 < r_{\mathcal{F}}(x),$$

we have $x \notin P \setminus \{y\}$. Therefore, $y = x$. $\square$

We shall need the following property of an isometry $\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$:

$$P \cap Q \subseteq R \subseteq P \cup Q \quad \Leftrightarrow \quad \alpha(P) \cap \alpha(Q) \subseteq \alpha(R) \subseteq \alpha(P) \cup \alpha(Q) \tag{5.3}$$

for $P, Q, R \in \mathcal{F}$. This property is an immediate consequence of equivalence of lattice and metric betweenness relations.

Let $\mathcal{F}$ and $\mathcal{G}$ be two wg-families each containing $\varnothing$ and $\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$ be an isometry such that $\alpha(\varnothing) = \varnothing$. Following Theorem 4.1, we need to show that there is a permutation $\pi : X \to X$ such that $\alpha = \hat{\pi}|_{\mathcal{F}}$.

As a special case of (5.3), we have

$$P \subseteq Q \quad \Leftrightarrow \quad \alpha(P) \subseteq \alpha(Q) \tag{5.4}$$

for $P, Q \in \mathcal{F}$, since P lies between $\varnothing$ and Q.

We also have

$$|\alpha(P)| = |P| \quad \text{for } P \in \mathcal{F}, \tag{5.5}$$

since $|P| = d(\varnothing, P) = d(\varnothing, \alpha(P)) = |\alpha(P)|$.

Lemma 5.3. *If $x \in P \in \mathcal{F}$ and $P \setminus \{x\} \in \mathcal{F}$, then there is $y \in \alpha(P)$ such that $\alpha(P) \setminus \{y\} = \alpha(P \setminus \{x\})$.*

Proof. By (5.4), $P \setminus \{x\} \subset P$ implies $\alpha(P \setminus \{x\}) \subset \alpha(P)$. Since $d(P \setminus \{x\}, P) = 1$, we have $d(\alpha(P \setminus \{x\}), \alpha(P)) = 1$. The result follows. $\square$

We define a relation $\pi \subseteq D(\mathcal{F}) \times D(\mathcal{G})$ as follows: $(x, y) \in \pi$ if and only if $x \in D(\mathcal{F})$ and $y \in D(\mathcal{G})$ satisfy conditions of Lemma 5.3 for some $P \in \mathcal{F}$.

By lemmas 5.2 and 5.3, for any $x \in D(\mathcal{F})$ there is $y \in D(\mathcal{G})$ such that $(x, y) \in \pi$. Conversely, for any $y \in D(\mathcal{G})$ there is $x \in D(\mathcal{F})$ such that $(x, y) \in \pi$. Indeed, it suffices to apply the results of lemmas 5.2 and 5.3 to the family $\mathcal{G}$ and the inverse isometry α^{-1}.

Lemma 5.4. *If $x \in X_k^{\mathcal{F}}$ and $(x, y) \in \pi$, then $y \in X_k^{\mathcal{G}}$. Conversely, if $y \in X_k^{\mathcal{G}}$ and $(x, y) \in \pi$, then $x \in X_k^{\mathcal{F}}$.*

Proof. Let $P \in \mathcal{F}$ be a set of cardinality k defining $r_{\mathcal{F}}(x) = k$. Then $r_{\mathcal{G}}(y) \leq k$, since $y \in \alpha(P)$ and, by (5.5), $|\alpha(P)| = k$.

Suppose that $m = r_{\mathcal{G}}(y) < k$. Then there is $Q \in \mathcal{G}$ such that $y \in Q$ and $|Q| = m$. By Lemma 5.2, $Q \setminus \{y\} \in \mathcal{G}$. By Lemma 5.3,

$$\alpha(P \setminus \{x\}) \cap Q \subseteq \alpha(P) \subseteq \alpha(P \setminus \{x\}) \cup Q.$$

By (5.3), we have

$$(P \setminus \{x\}) \cap \alpha^{-1}(Q) \subseteq P \subseteq (P \setminus \{x\}) \cup \alpha^{-1}(Q).$$

Thus, $x \in \alpha^{-1}(Q)$, a contradiction, since $r_{\mathcal{F}}(x) = k$ and, by (5.5),

$$|\alpha^{-1}(Q)| = |Q| = m < k.$$

It follows that $r_{\mathcal{G}}(y) = k$, that is, $y \in X_k^{\mathcal{G}}$.

We prove the converse statement by applying the above argument to the inverse isometry α^{-1}. ☐

We proved that for every $k \geq 1$ the restriction of π to $X_k^{\mathcal{F}}$ is a relation $\pi_k \subseteq X_k^{\mathcal{F}} \times X_k^{\mathcal{G}}$.

Lemma 5.5. *The relation π_k is a bijection for every $k \geq 1$.*

Proof. First we prove that π_k is a function. Suppose that there are $z \neq y$ such that $(x, y) \in \pi_k$ and $(x, z) \in \pi_k$. Then there are two distinct sets $P, Q \in \mathcal{F}$ defining y and z, respectively, such that

$$x \in P \cap Q, \;\; k = r_{\mathcal{F}}(x) = |P| = |Q|, \;\; P \setminus \{x\} \in \mathcal{F}, \; Q \setminus \{x\} \in \mathcal{F}.$$

By Lemma 5.3,

$$\alpha(P) \setminus \{y\} = \alpha(P \setminus \{x\}), \;\; \alpha(Q) \setminus \{z\} = \alpha(Q \setminus \{x\})$$

for some $y \in \alpha(P)$ and $z \in \alpha(Q)$. We have

$$\begin{aligned} d\,(\alpha(P), \alpha(Q)) &= d\,(P, Q) = d\,(P \setminus \{x\}, Q \setminus \{x\}) \\ &= d\,(\alpha(P) \setminus \{y\}, \alpha(Q) \setminus \{z\}). \end{aligned}$$

Thus, $y, z \in \alpha(P) \cap \alpha(Q)$. In particular, $z \in \alpha(P) \setminus \{y\}$, a contradiction, because $|\alpha(P) \setminus \{y\}| = k - 1$ but, by Lemma 5.4, $r_{\mathcal{G}}(z) = k$.

By applying the above argument to α^{-1}, we prove that for any $y \in X_k^{\mathcal{G}}$ there is a unique $x \in X_k^{\mathcal{F}}$ such that $(x, y) \in \pi_k$. Hence, π_k is a bijection. ☐

Corollary 5.1. *The relation π is a bijection from $D(\mathcal{F}_1)$ onto $D(\mathcal{F}_2)$.*

By condition (5.1), the bijection π can be extended to a permutation on the set X. We denote this permutation by the same symbol π.

Lemma 5.6. $\alpha(P) = \pi(P)$ *for any* $P \in \mathcal{F}_1$.

Proof. The proof is by induction on $k = |P|$. The case $k = 1$ is trivial, since $\alpha(\{x\}) = \{\pi_1(x)\}$ for $\{x\} \in \mathcal{F}$.

Suppose that $\alpha(R) = \pi(P)$ for all $R \in \mathcal{F}$ such that $|R| < k$. Let P be a set in $\mathcal{F}$ of cardinality k. By Lemma 5.1, $P = R \cup \{x\}$ for some $R \in \mathcal{F}$ and $x \notin R$. It is clear that $m = r_{\mathcal{F}}(x) \leq k$ and $|R| = k - 1$.

If $m = k$, then $\alpha(P) = \alpha(R) \cup \{\pi(x)\} = \pi(P)$, by the definition of π and the induction hypothesis.

Suppose that $m < k$. Since $m = r_{\mathcal{F}}(x)$, there is a set $Q \in \mathcal{F}$ containing x such that $|Q| = m$. By Lemma 5.2, there is $S \in \mathcal{F}$ such that $S = Q \setminus \{x\}$. Since $x \in P$, we have

$$S \cap P \subseteq Q \subseteq S \cup P.$$

By (5.3), we have

$$\alpha(S) \cap \alpha(P) \subseteq \alpha(Q) \subseteq \alpha(S) \cup \alpha(P).$$

Thus, by the induction hypothesis,

$$\pi(S) \cup \{\pi(x)\} = \pi(Q) \subseteq \pi(S) \cup \alpha(P).$$

Since $\pi(x) \notin \pi(S)$, we have $\pi(x) \in \alpha(P)$. Since $\alpha(P) = \pi(R) \cup \{y\}$ for $y \notin \pi(R)$, and $x \notin R$, we have $y = \pi(x)$, that is, $\alpha(P) = \pi(P)$. $\square$

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