MATH 710, Fall 2009, Exam #2, Solutions

1. Let h be a measurable function on $(0,1)$ such that $|h| < M$ for some $M > 0$ and $h \le f$ on $(0,1)$. Let us define f_M on $(0,1)$ by

$$f_M(x) = \min\{f(x), M\} = \begin{cases} f(x), & \text{if } 0 < x < 1 - \frac{1}{M^3}, \\ M, & \text{if } 1 - \frac{1}{M^3} \le x < 1. \end{cases}$$

Clearly, $h \le f_M < f$ on $(0,1)$ and f_M is measurable. We have

$$\int_0^1 f_M = \int_0^{1-\frac{1}{M^3}} f_M + \int_{1-\frac{1}{M^3}}^1 f_M = \int_0^{1-\frac{1}{M^3}} \frac{1}{\sqrt[3]{1-x}}\,dx + \int_{1-\frac{1}{M^3}}^1 M\,dx$$
$$= -\frac{3}{2}(1-x)^{2/3}\Big|_0^{1-\frac{1}{M^3}} + M \cdot \frac{1}{M^3} = -\frac{3}{2M^2} + \frac{3}{2} + \frac{1}{M^2} = \frac{3}{2} - \frac{1}{2M^2}$$

Since $h \le f_M$, we have $\int_0^1 h < 3/2$ for any bounded $h \le f$. Therefore, f is Lebesgue integrable and

$$\int_0^1 \frac{1}{\sqrt[3]{1-x}}\,dx = \sup_{h \le f} \int_0^1 h = \frac{3}{2}$$

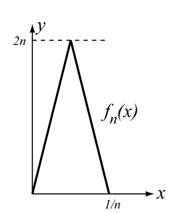
2. Consider the sequence of simple functions

$$f_n(x) = \begin{cases} f(x), & x \in \cup_{k=0}^n E_k, \\ 0, & \text{otherwise} \end{cases}$$

It is bounded, $|f_n| \le 1$, and converges to f. By Proposition 6, we have

$$\int_E f_n = \sum_{k=0}^{n+1} \frac{1}{2^{k+1}}(-1)^k \to \frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3} = \int_E f$$

3. False. Let $\langle f_n \rangle$ be the sequence of functions defined on $[0,1]$ shown in the drawing below where $f_n(x) = 0$ for $x \in [1/n, 1]$:



Clearly, $f_n(x) \to 0$ for all $x \in [0,1]$. However, $\int_0^1 f_n(x)\,dx = 1$ for all $n \in \mathbb{N}$.

4. False. Consider $f(x) = \frac{1}{x}$ on $(0,1]$. Let $E_k = (\frac{1}{k+1}, \frac{1}{k}]$, so $\cup_1^\infty E_k = (0,1]$. Clearly, f is integrable over each E_k with $\int_{E_k} f = \ln(k+1) - \ln k$. However, f is not integrable over $(0,1]$. Indeed, let $h_n = \min\{f(x), n\}$ on $(0,1]$, so $h_n < f$. We have

$$\int_0^1 h_n\,dx = \int_0^{1/n} n\,dx + \int_{1/n}^1 \frac{dx}{x} = 1 + \ln n$$

Hence,

$$\int_0^1 \frac{dx}{x} = \sup_{h \le f} \int_0^1 h\,dx = \infty$$