MATH 710 – FALL 2011

EXAMINATION #1

1. Let A be a closed subset of $[0,1]$ such that $m(A) = 1$. Prove that $A = [0,1]$.

2. Show that a set E is measurable if and only if for each $\varepsilon > 0$, there is a closed set F and an open set G for which $F \subseteq E \subseteq G$ and $m^*(G \setminus F) < \varepsilon$.

3. Prove that a function f on $[a,b]$ is measurable if and only if $f^{-1}(U)$ is measurable for any open set $U \subseteq \mathbb{R}$.

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1. Let $a = \inf A$, $b = \sup A$ (cf. Theorem 1.15 and Definition 2.9). We have

$$1 = m(A) \leq b - a,$$

which implies $a = 0$, $b = 1$, because $a \geq 0$ and $b \leq 1$. The set $B = [0,1] \setminus A$ is open. If $B \neq \varnothing$, then $m(B) > 0$, which contradicts $m(A) = 1$.

2. (Necessity.) Suppose E is measurable and let $\varepsilon > 0$. By Exercises 2.13 and 2.4, there are an open set $G \supseteq E$ and a closed set $F \subseteq E$ such that

$$m(G) < m(E) + \varepsilon/2 \quad \text{and} \quad m(F) > m(E) - \varepsilon/2.$$

It follows that

$$m(G \setminus F) = m(G) - m(F) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(Sufficiency.) Let E be a bounded set such that there are a closed set F and an open set G for which $F \subseteq E \subseteq G$ and $m^*(G \setminus F) < \varepsilon$, for every $\varepsilon > 0$. We have

$$m(F) = m_*(F) \leq m_*(E) \leq m^*(E) \leq m^*(G) = m(G).$$

Because sets F, G, and $G \setminus F$ are measurable, we have

$$\varepsilon > m^*(G \setminus F) = m(G \setminus F) = m(G) - m(F).$$

Therefore, $0 \leq m^*(E) - m_*(E) < \varepsilon$ for an arbitrary positive ε. It follows that $m^*(E) = m_*(E)$, that is, E is measurable.

3. (Necessity.) Let f be a measurable function on $[a,b]$ and U be an open subset of $\mathbb{R}$. We have

$$U = \bigcup_{i \in J} (a_i, b_i), \qquad \text{where } J \text{ is at most countable set.}$$

Therefore,

$$f^{-1}(U) = \bigcup_{i \in J} f^{-1}((a_i, b_i)).$$

Each set $f^{-1}((a_i, b_i))$ is measurable (cf. Theorem 2.48). Hence $f^{-1}(U)$ is measurable.

(Sufficiency.) Since $(c, +\infty)$ is an open set for any real c, the set

$$\{x \in [a,b] : f(x) > c\}$$

is measurable. Hence, f is measurable.