

MATH 710 – FALL 2009, **Assignment #2, Solutions**

p.58, #5. Let  $A$  be the set of rational numbers between 0 and 1, and let  $\{I_n\}$  be a *finite* collection of open intervals covering  $A$ . Then  $\sum \ell(I_n) \geq 1$ .

*Solution 1.* We have  $A \subset \cup_n I_n$ . Take the closure on both sides of this inclusion. Since  $\bar{A} = [0, 1]$  and the family  $\{I_n\}$  is finite, we have

$$[0, 1] = \bar{A} \subset \overline{\cup_n I_n} = \cup_n \bar{I}_n$$

Therefore,

$$1 \leq m^*(\cup_n \bar{I}_n) \leq \sum_n \ell(I_n)$$

since  $\ell(\bar{I}_n) = \ell(I_n)$ .

*Solution 2.* The set  $O = \cup_n I_n$  is open. Therefore, it is a union of disjoint open intervals. This union is finite (easy proof by induction). Hence we have

$$A \subset O = (a_1, b_1) \cup \cdots \cup (a_m, b_m)$$

with  $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ . We may assume that  $0 \in (a_1, b_1)$  and  $1 \in (a_m, b_m)$ . (Justify it!) Since any nontrivial subinterval of  $(0, 1)$  contains rational numbers and  $O$  covers  $A$ , we must have

$$a_1 < b_1 = a_2 < b_2 = a_3 < \cdots = a_m < b_m$$

Hence,  $1 \leq b_m - a_1 = m^*O \leq \sum_n \ell(I_n)$ .

p.58, #7. Prove that  $m^*$  is translation invariant.

*Solution.* We need to prove that for any  $A \subset \mathbb{R}$  and  $a \in \mathbb{R}$ , we have  $m^*A = m^*(A+a)$ . Clearly,  $\ell(I+a) = \ell(I)$ , for any interval. Since  $A \subset \cup_n I_n$  implies  $A+a \subset \cup_n (I_n+a)$ , we have

$$\begin{aligned} m^*A &= \inf_{A \subset \cup_n I_n} \sum \ell(I_n) = \inf_{A+a \subset \cup_n (I_n+a)} \sum \ell(I_n+a) \geq \\ &\geq \inf_{A+a \subset \cup_n J_n} \sum \ell(J_n) = m^*(A+a) \end{aligned}$$

By applying the above inequality to  $A+a$  and  $-a$ , we obtain  $m^*(A+a) \geq m^*A$ . The result follows.

p.58, #8. Prove that if  $m^*A = 0$ , then  $m^*(A \cup B) = m^*B$ .

*Solution.* Since  $B \subset A \cup B$ , we have, by Prop. 2 and monotonicity of  $m^*$ ,

$$m^*B \leq m^*(A \cup B) \leq m^*A + m^*B = m^*B$$

Hence,  $m^*(A \cup B) = m^*B$ .