Extended Appendix to "Exchange Rates Under Robustness: An Account of the Forward Premium Puzzle" Li and Tornell (2008)

Here, we present some proofs as well as some auxiliary results.

Proof of Lemma 6.1. We prove part 1 in three steps. First, we find the distribution of random variable $x_t|x_{t-1}$ under any probability measure $\theta \in \Theta^w$, given that under the baseline measure θ' the random variable $x_t|x_{t-1}$ is normally distributed as $N\left(ax_{t-1}, \sigma_w^2\right)$. Second, we show that $y_t|x_t$ and x_{t-1} have the same distribution under measure θ as under measure θ'.

$P^{\theta}\left(x_t < z|x_{t-1}\right) \equiv \int_{\{x_t<z|x_{t-1}\}} d\theta$

$$
\begin{aligned}
&= \int_{\{x_t<z|x_{t-1}\}} \frac{d\theta}{d\theta'} d\theta' \\
&= \int_{\{x_t<z|x_{t-1}\}} \exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2} - \frac{1}{2\sigma_w^2}\right)\left(x_t - ax_{t-1}\right)^2\right) \cdot \sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}} d\theta' \\
&= \int_{\infty}^{z} \exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2} - \frac{1}{2\sigma_w^2}\right)\left(x - ax_{t-1}\right)^2\right) \cdot \sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}} \frac{1}{\sqrt{2\pi}\sigma_w} \exp\left[-\frac{\left(x - ax_{t-1}\right)^2}{2\sigma_w^2}\right] dx \\
&= \int_{\infty}^{z} \frac{1}{\sqrt{2\pi}\tilde{\sigma}_w} \exp\left[-\frac{1}{2\tilde{\sigma}_w^2}\left(x - ax_{t-1}\right)^2\right] dx
\end{aligned}
$$

The RHS in the last equation is the PDF of a Normal distribution $N\left(ax_{t-1}, \tilde{\sigma}_w^2\right)$. This shows that $x \overset{\theta}{\sim} N\left(ax_{t-1}, \tilde{\sigma}_w^2\right)$. Second, we show that $y_t|x_t$ has the same distribution under measure θ as under measure θ'.

$$
\begin{aligned}
P^{\theta}\left(y_t < z|x_t\right) &= E^{\theta} 1_{\{y_t<z|x_t\}} = E^{\theta'} 1_{\{y_t<z|x_t\}} \frac{d\theta}{d\theta'} \\
&= E^{\theta'} 1_{\{y_t<z|x_t\}} \exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2} - \frac{1}{2\sigma_w^2}\right) w_{t-1}^2\right) \cdot \sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}} \\
&= E^{\theta'} \left\{1_{\{y_t<z|x_t\}} \exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2} - \frac{1}{2\sigma_w^2}\right) w_{t-1}^2\right) \cdot \sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}} |x_t\right\} \\
&= E^{\theta'} \left\{E^{\theta'} \left\{1_{\{y_t<z|x_t\}} \exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2} - \frac{1}{2\sigma_w^2}\right) w_{t-1}^2\right) \cdot \sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}} |x_{t-1}\right\} |x_t\right\} \\
&= E^{\theta'} \left\{E^{\theta'} \left\{1_{\{y_t<z\}}|x_{t-1}\right\} E^{\theta'} \left\{\exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2} - \frac{1}{2\sigma_w^2}\right) w_{t-1}^2\right) \cdot \sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}}\right\} |x_t\right\}
\end{aligned}
$$

The last inequality follows because x_{t-1} and w_{t-1} are independent with each other under

θ'. Notice that the third expectation in the last inequality equals one because

$$
\begin{aligned}
& E^{\theta'}\left\{\exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2}-\frac{1}{2\sigma_w^2}\right)w_{t-1}^2\right)\cdot\sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}}\right\} \\
& \int \exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2}-\frac{1}{2\sigma_w^2}\right)w_{t-1}^2\right)\cdot\sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}}d\theta' \\
= & \int \exp\left(-\left(\frac{1}{2\tilde{\sigma}_w^2}-\frac{1}{2\sigma_w^2}\right)w_{t-1}^2\right)\cdot\sqrt{\frac{\sigma_w^2}{\tilde{\sigma}_w^2}}\frac{1}{\sqrt{2\pi\sigma_w^2}}\exp\left(-\frac{1}{2}\frac{w_{t-1}^2}{\sigma_w^2}\right)dw_{t-1} \\
= & \int \frac{1}{\sqrt{2\pi\tilde{\sigma}_w^2}}\exp\left(-\frac{1}{2}\frac{w_{t-1}^2}{\tilde{\sigma}_w^2}\right)dw_{t-1}=1
\end{aligned}
$$

Therefore, we have that $P^{\theta}\left(y_t<z|x_t\right)=E^{\theta}1_{\{y_t<z\}}=E^{\theta'}1_{\{y_t<z\}}$. This shows that conditional on x_t, y_t has the same distribution under θ as under θ'. Third, the same argument can be used to show that x_{t-1} has the same distribution under θ as under θ'. To prove part 2 we compute the relative entropy

$$
\begin{aligned}
R\left(\theta||\theta'\right) &= E^{\theta}\log\left(\frac{d\theta}{d\theta'}\right)=E^{\theta}\left(-\left(\frac{1}{2\tilde{\sigma}_w^2}-\frac{1}{2\sigma_w^2}\right)x^2-\frac{1}{2}\log\left(\tilde{\sigma}_w^2\right)\right) \\
&= -\left(\frac{1}{2\tilde{\sigma}_w^2}-\frac{1}{2\sigma_w^2}\right)\cdot E^{\theta}\left(x^2\right)-\frac{1}{2}\log\left(\tilde{\sigma}_w^2\right) \\
&= \frac{1}{2}\left(\frac{\tilde{\sigma}_w^2}{\sigma_w^2}-\log\left(\frac{\tilde{\sigma}_w^2}{\sigma_w^2}\right)-1\right)
\end{aligned}
$$

Proof of Proposition 6.2. The solution to the agent's problem has the same form as in the case of observation uncertainty. The first order condition for b_t is given by (8.3), while that for $\tilde{\sigma}_{w,t}^2$ is

$$
\begin{aligned}
0 &= \frac{\partial\Gamma}{\partial\tilde{\sigma}_{w,t}^2} \\
&= -\frac{1}{2}\left(\gamma b_t\right)^2\frac{\partial Var^{\theta_t}\left(J_{t+1}\right)}{\partial\tilde{\sigma}_{w,t}^2}\exp\left(-\gamma b_t E_t^{\theta_t}\left(J_{t+1}\right)+\frac{1}{2}\left(\gamma b_t\right)^2 Var_t^{\theta_t}\left(J_{t+1}\right)\right)+\frac{\lambda}{2}\left(\frac{1}{\sigma_w^2}-\frac{1}{\tilde{\sigma}_{w,t}^2}\right)
\end{aligned}
$$

where

$$
\frac{\partial Var_t^{\theta_t}\left(J_{t+1}\right)}{\partial\tilde{\sigma}_{w,t}^2}=\left[\left(k_{t+1}^{\theta_{t+1}}\beta_1+\beta_2\right)\right]^2\left(a^2\frac{\sigma_v^2}{\left(a^2\sigma_{t-1}^2+\sigma_v^2+\tilde{\sigma}_{w,t}^2\right)^2}+1\right)>0
$$

Analogous to the argument for $\tilde{\sigma}_{v,t}^{2\theta^*}$, the first order conditions imply that a robust agent distorts upwards the variance of the trend shock: $\tilde{\sigma}_{w,t}^{2*}>\sigma_w^2$. The second order condition for b_t is satisfied because

$$\begin{aligned}\frac{\partial^2\Gamma}{\partial b_t^2} &= -\left[\left(-\gamma E_t^{\theta_t}\left(J_{t+1}\right)+\left(\gamma^2 b_t\right)Var_t^{\theta}\left(J_{t+1}\right)\right)^2+\gamma^2 Var_t^{\theta}\left(J_{t+1}\right)\right]\cdot \\ &\quad \exp\left(-\left(\gamma b_t\right)E_t^{\theta_t}\left(J_{t+1}\right)+\frac{1}{2}\left(\gamma b_t\right)^2 Var_t^{\theta_t}\left(J_{t+1}\right)\right) \\ &< 0\end{aligned}$$

The second derivative of $\tilde{\sigma}_{w,t}^2$ is

$$\begin{aligned}\frac{\partial^2\Gamma}{\partial\left(\tilde{\sigma}_{w,t}^2\right)^2} &= -\frac{\left(\gamma b_t\right)^2}{2}\left[\frac{1}{2}\left(\gamma b_t\right)^2\left(\frac{\partial Var_t^{\theta_t}\left(J_{t+1}\right)}{\partial\tilde{\sigma}_{w,t}^2}\right)^2+\frac{\partial^2 Var_t^{\theta_t}\left(J_{t+1}\right)}{\partial\left(\tilde{\sigma}_{w,t}^2\right)^2}\right] \\ &\quad \cdot\exp\left(-\gamma b_t E_t^{\theta_t}\left(J_{t+1}\right)+\frac{\left(\gamma b_t\right)^2}{2}Var_t^{\theta_t}\left(J_{t+1}\right)\right)+\frac{\lambda}{2}\frac{1}{\left(\tilde{\sigma}_{w,t}^2\right)^2}\geq 0\end{aligned}$$

where $\frac{\partial^2 Var_t^{\theta_t}\left(J_{t+1}\right)}{\partial\left(\tilde{\sigma}_{w,t}^2\right)^2}=-\left[\left(k_{t+1}^{\theta_{t+1}}\beta_1+\beta_2\right)\right]^2\frac{a^2\sigma_v^2}{\left(a^2\sigma_{t-1}^2+\sigma_v^2+\tilde{\sigma}_{w,t}^2\right)^3}$. Hence the second order condition for $\tilde{\sigma}_{w,t}^2$ holds if and only if $\lambda\geq\lambda_t^w$, where λ_t^w is defined by (8.18).

Proof of Lemma 6.3. We follow the same steps as in the proof of Lemma 6.1. First, we find the distribution of random variable $x_t|x_{t-1}$ under any probability measure $\theta\in\Theta^a$, given that under the baseline measure θ' the random variable $x_t|x_{t-1}$ has a normal distribution $N\left(ax_{t-1},\sigma_w^2\right)$. Then we prove that $y_t|x_t$ and x_{t-1} have the same distributions under θ as under θ'.

$$\begin{aligned}P^{\theta}\left(x_t<z|x_{t-1}\right) &\equiv \int_{\{x_t<z|x_{t-1}\}}d\theta=\int_{\{x_t<z|x_{t-1}\}}\frac{d\theta}{d\theta'}d\theta' \\ &= \int_{\{x_t<z|x_{t-1}\}}\exp\left[-\frac{\left(x_{t-1}\delta\right)^2-2x_{t-1}\delta\left(x_t-ax_{t-1}\right)}{2\sigma_w^2}\right]d\theta' \\ &= \int_{-\infty}^{z}\exp\left[-\frac{\left(x_{t-1}\delta\right)^2-2x_{t-1}\delta\left(x-ax_{t-1}\right)}{2\sigma_w^2}\right]\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\left(x-ax_{t-1}\right)^2}{2\sigma_w^2}\right]dx \\ &= \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma_w^2}\left(x-\left(a+\delta\right)x_{t-1}\right)^2\right]dx\end{aligned}$$

The RHS in the last equation is the PDF of a Normal distribution $N\left(\left(a+\delta\right)x_{t-1},\sigma_w^2\right)$. This shows that $x_t\overset{\theta}{\backsim}N\left(\left(a+\delta\right)x_{t-1},\sigma_w^2\right)$. The same argument as the one in the proof of Lemma 6.1 can be applied to prove that $y_t|x_t$ and x_{t-1} have the same distributions under θ as under θ'. To prove part 2, we compute the relative entropy

$$\begin{aligned} R(\theta||\theta') &= E^{\theta}\log\left(\frac{d\theta}{d\theta'}\right) \\ &= E^{\theta}\left(-\frac{(x_{t-1}\delta)^2 - 2x_{t-1}\delta\,(x_t - ax_{t-1})}{2\sigma_w^2}\right) = E^{\theta}\frac{(x_t\delta)^2}{2\sigma_w^2}. \end{aligned}$$

To derive the last equality we use the fact that $E^{\theta}(x_t|x_{t-1}) = (a+\delta)\,x_{t-1}$.

Proof of Proposition 6.4. Since the prior of the young t agent is $x_{t-1} \overset{\theta'}{\sim} N\left(x_{t-1}^{\theta_{t-1}}, \sigma_{t-1}^2\right)$, where $\sigma_{t-1}^2 = \frac{\left(a^2\sigma_{t-2}^2+\sigma_w^2\right)\sigma_v^2}{a^2\sigma_{t-2}^2+\sigma_w^2+\sigma_v^2}$, it follows that under the robust model θ_t, the agent's posterior estimate of the state x_t is

$$\begin{aligned} &\hat{x}_t^{\theta_t} = E_t^{\theta_t}(x_t|I_t) = \left(1-k_t^{\theta_t}\right)(a+\delta_t)\,\hat{x}_{t-1}^{\theta_{t-1}} + k_t^{\theta_t}y_t \\ &x_t|I_t \overset{\theta_t}{\backsim} N\left(\hat{x}_t^{\theta_t}, \frac{\left((a+\delta_t)^2\sigma_{t-1}^2+\sigma_w^2\right)\sigma_v^2}{(a+\delta_t)^2\sigma_{t-1}^2+\sigma_w^2+\sigma_v^2}\right), \\ &x_{t+1}|I_t \overset{\theta_t}{\backsim} N\left((a+\delta_t)\,\hat{x}_t, (a+\delta_t)^2\frac{\left((a+\delta_t)^2\sigma_{t-1}^2+\sigma_w^2\right)\sigma_v^2}{(a+\delta_t)^2\sigma_{t-1}^2+\sigma_w^2+\sigma_v^2} + \sigma_w^2\right) \end{aligned}$$

In this case the gain of the filter is $k_t^{\theta_t} = \frac{(a+\delta_t)^2\sigma_{t-1}^2+\sigma_w^2}{(a+\delta_t)^2\sigma_{t-1}^2+\sigma_w^2+\sigma_v^2}$. Note that the gain is increasing in the drift distortion because of the assumption $\delta_t \geq -a$

$$\frac{dk_t}{d\delta_t} = \frac{2\,(a+\delta_t)\,\sigma_{t-1}^2}{\left((a+\delta_t)^2\sigma_{t-1}^2+\sigma_w^2+\sigma_v^2\right)^2} > 0 \tag{8.23}$$

Under probability measure θ_t the log excess return is

$$\begin{aligned} J_{t+1} &\equiv (i_t - i_t^f) - (e_{t+1} - e_t) \\ &= -\alpha_{t+1} - \beta_1\hat{x}_{t+1}^{\theta_{t+1}} - \beta_2\left(i_t - i_{t+1}^f\right) + e_t + (i_t - i_t^f) \\ &= -\alpha_{t+1} - \left(1-k_{t+1}^{\theta_{t+1}}\right)\beta_1\,(a+\delta_{t+1})\,\hat{x}_t^{\theta_t} - \left(k_{t+1}^{\theta_{t+1}}\beta_1 + \beta_2\right)\left(i_t - i_{t+1}^f\right) + e_t + (i_t - i_t^f) \end{aligned}$$

To compute the distribution of excess returns, under probability measure θ_t, the t agent uses the conjecture (3.7) to forecast next period's exchange rate: $e_{t+1}^{conj} = \alpha_{t+1} + \beta_1\hat{x}_{t+1}^{\theta_{t+1}} + \beta_2\left(i_{t+1} - i_{t+1}^f\right)$. To obtain $E_t^{\theta_t}(\hat{x}_{t+1}^{\theta_{t+1}}; I_t)$ note that the t agent knows the problem that will be solved by $t+1$ agents. Thus, the t agent knows the method that $t+1$ agents will use to derive the robust probability measure θ_{t+1}, and that they will make forecasts using Bayes law under θ_{t+1}. Taking this into account, the t agent knows that $t+1$ agents will use the updating formula $\hat{x}_{t+1}^{\theta_{t+1}} = \left(1-k_{t+1}^{\theta_{t+1}}\right)(a+\delta_{t+1})\hat{x}_t^{\theta_t} + k_{t+1}^{\theta_{t+1}}\left(i_{t+1} - i_{t+1}^f\right)$. Therefore, the t agent sets $E_t^{\theta_t}(\hat{x}_{t+1}^{\theta_{t+1}}) = (a+\delta_{t+1})\hat{x}_t^{\theta_t}$. Replacing this formula in the conjecture, it follows that under probability measure θ_t, the log excess return J_{t+1}

$$\begin{aligned}
E_t^{\theta_t}\left(J_{t+1}\right) &= -\alpha_{t+1} - \left(1 - k_{t+1}^{\theta_{t+1}}\right)\beta_1\left(a+\delta_{t+1}\right)\hat{x}_t^{\theta_t} - \left(k_{t+1}^{\theta_{t+1}}\beta_1+\beta_2\right)\left(a+\delta_t\right)\hat{x}_t^{\theta_t} + e_t + \left(i_t - i_t^f\right) \\
&= -\alpha_{t+1} - \left(\left(1 - k_{t+1}^{\theta_{t+1}}\right)\beta_1\left(a+\delta_{t+1}\right) + \left(k_{t+1}^{\theta_{t+1}}\beta_1+\beta_2\right)\left(a+\delta_t\right)\right)\hat{x}_t^{\theta_t} + e_t + \left(i_t - i_t^f\right) \\
V_t^{\theta_t}\left(J_{t+1}\right) &= \left(k_{t+1}^{\theta_{t+1}}\beta_1+\beta_2\right)^2\left[\left(a+\delta_t\right)^2\frac{\left(\left(a+\delta_t\right)^2\sigma_{t-1}^2+\sigma_w^2\right)\sigma_v^2}{\left(a+\delta_t\right)^2\sigma_{t-1}^2+\sigma_w^2+\sigma_v^2}+\sigma_w^2+\sigma_v^2\right]
\end{aligned}$$

We solve problem (3.4) by considering the investor as a Stackelberg leader that takes into account the strategy of nature: $\delta_t = s(b_t, e_t)$. Nature then selects δ_t conditioning on the agent's choice of b_t. Notice that δ_t affects the investor's payoff through its effect on $E_t^{\theta_t}\left(J_{t+1}\right)$ and $Var_t^{\theta_t}\left(J_{t+1}\right)$. The first order condition for δ_t is:

$$\begin{aligned}
\frac{\partial\Gamma}{\partial\delta_t} &= -\left[\gamma b_t\frac{\partial E_t^{\theta_t}\left(J_{t+1}\right)}{\partial\delta_t}+\frac{\left(\gamma b_t\right)^2}{2}\frac{\partial Var_t^{\theta_t}\left(J_{t+1}\right)}{\partial\delta_t}\right]\exp\left(-\gamma b_t E_t^{\theta_t}\left(J_{t+1}\right)+\frac{\left(\gamma b_t\right)^2}{2}Var_t^{\theta_t}\left(J_{t+1}\right)\right) \\
&\quad +\lambda\frac{E_t^{\theta_t}x_t^2}{\sigma_w^2}\delta_t \\
&= -\left[-\gamma b_t\left(k_{t+1}^{\theta_{t+1}}\beta_1+\beta_2\right)\hat{x}_t^{\theta_t}+\frac{\left(\gamma b_t\right)^2}{2}\left(k_{t+1}\beta_1+\beta_2\right)^2\left(2\left(a+\delta_t\right)k_t^{\theta_t}+\left(a+\delta_t\right)^2\frac{dk_t}{d\delta_t}\right)\right]\times \\
&\quad \exp\left(-\gamma b_t E_t^{\theta_t}\left(J_{t+1}\right)+\frac{\left(\gamma b_t\right)^2}{2}Var_t^{\theta_t}\left(J_{t+1}\right)\right)+\lambda\frac{E_t^{\theta_t}x_t^2}{\sigma_w^2}\delta_t
\end{aligned}$$

The first order condition for for b_t is:

$$\begin{aligned}
\frac{\partial\Gamma}{\partial b_t} &= -\left(-\gamma E_t^{\theta_t}\left(J_{t+1}\right)+\gamma^2 b_t Var_t^{\theta_t}\left(J_{t+1}\right)\right)\exp\left(-\gamma b_t E_t^{\theta_t}\left(J_{t+1}\right)+\frac{\left(\gamma b_t\right)^2}{2}Var_t^{\theta_t}\left(J_{t+1}\right)\right) \\
&\quad +\frac{\partial\Gamma}{\partial\delta_t}\frac{d\delta_t}{db_t} \\
&= \left(\gamma E_t^{\theta_t}\left(J_{t+1}\right)-\gamma^2 b_t Var_t^{\theta_t}\left(J_{t+1}\right)\right)\exp\left(-\gamma b_t E_t^{\theta_t}\left(J_{t+1}\right)+\frac{\left(\gamma b_t\right)^2}{2}Var_t^{\theta_t}\left(J_{t+1}\right)\right)
\end{aligned}$$

In the second equality we have used the first order condition for nature: $\frac{\partial\Gamma}{\partial\delta_t}=0$. The second order condition for the investor's problem is

$$\begin{aligned}
0 &> \frac{\partial^2\Gamma}{\partial b_t^2} = \Gamma_{b_tb_t}(b_t,\delta_t^*(b_t)) + \Gamma_{b_t\delta_t}(b_t,\delta_t^*(b_t))\frac{d\delta_t^*(b_t)}{db_t} \\
&\quad +\Gamma_{\delta_t\delta_t}(b_t,\delta_t^*(b_t))\cdot\left(\frac{d\delta_t^*(b_t)}{db_t}\right)^2+\Gamma_{\delta_t}(b_t,\delta_t^*(b_t))\cdot\frac{d^2\delta_t^*(b_t)}{db_t^2}
\end{aligned}$$

Notice that the total derivative of nature's FOC $\Gamma_{\delta_t}(b_t,\delta_t^*(b_t))=0$ is $\Gamma_{\delta_tb_t}db_t+\Gamma_{\delta_t}d\delta_t=0$.

Thus,

$$\frac{d\delta_t^*(b_t)}{db_t} = -\frac{\Gamma_{\delta_t b_t}}{\Gamma_{\delta_t\delta_t}}$$

Combining this equation with $\Gamma_{b_t\delta_t} = \Gamma_{\delta_t b_t}$, the investor's SOC becomes

$$\begin{aligned}\frac{\partial^2\Gamma}{\partial b_t^2} &= \Gamma_{b_t b_t}(b_t,\delta_t^*(b_t)) + \Gamma_{b_t\delta_t}(b_t,\delta_t^*(b_t)) \cdot \left(-\frac{\Gamma_{\delta_t b_t}(b_t,\delta_t^*(b_t))}{\Gamma_{\delta_t\delta_t}(b_t,\delta_t^*(b_t))}\right) + \\ &\quad \Gamma_{\delta_t\delta_t}(b_t,\delta_t^*(b_t)) \cdot \left(-\frac{\Gamma_{\delta_t b_t}(b_t,\delta_t^*(b_t))}{\Gamma_{\delta_t\delta_t}(b_t,\delta_t^*(b_t))}\right)^2 = \Gamma_{b_t b_t}(b_t,\delta_t^*(b_t)) \le 0\end{aligned}$$

This condition is unambiguously satisfied because $\Gamma_{b_t b_t} \le 0$:

$$\begin{aligned}\Gamma_{b_t b_t} &= -\left[\left[-\gamma E_t^{\theta_t}(J_{t+1}) + \left(\gamma^2 b_t\right) Var_t^{\theta_t}(J_{t+1})\right]^2 + \gamma^2 Var_t^{\theta_t}(J_{t+1})\right] \\ &\quad \cdot \exp\left(-(\gamma b_t) E_t^{\theta_t}(J_{t+1}) + \frac{1}{2}(\gamma b_t)^2 Var_t^{\theta_t}(J_{t+1})\right).\end{aligned}$$

Nature's second order condition is

$$\begin{aligned}\frac{\partial^2\Gamma}{\partial\delta_t^2} &= -\frac{(\gamma b_t)^2}{2}\left[\frac{1}{2}(\gamma b_t)^2\left(\frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial\delta_t}\right)^2 + \frac{\partial^2 Var_t^{\theta_t}(J_{t+1})}{\partial(\delta_t)^2}\right] \\ &\quad \cdot \exp\left(-\gamma b_t E_t^{\theta_{t+1}}(J_{t+1}) + \frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1})\right) + \lambda\frac{E_t^{\theta_t} x_t^2}{\sigma_w^2} \ge 0\end{aligned}$$

Hence, the second order condition for δ_t holds if and only if $\lambda \ge \lambda_t^a$, where λ_t^a is defined by

$$\lambda_t^a = \max\{0, \lambda_t^{***}\}, \quad \text{with} \tag{8.24}$$

$$\lambda_t^{***} \equiv \frac{(\gamma b_t)^2}{\left(\hat{x}_t^{\theta_t}\right)^2+1} \cdot \left[\frac{(\gamma b_t)^2}{2}\left(\frac{\partial Var_t^{\theta_t}(J_t)}{\partial\delta_t}\right)^2 + \frac{\partial^2 Var_t^{\theta_t}(J_{t+1})}{\partial\delta_t^2}\right] \cdot \exp\left(-\gamma b_t E_t^{\theta_t}(J_{t+1}) + \frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1})\right) \tag{8.25}$$

Next, we characterize the equilibrium. The first order condition for b_t implies that

$$\begin{aligned}&-\left[\alpha_{t+1} + \left(\left(1-k_{t+1}^{\theta_{t+1}}\right)\beta_1(a+\delta_{t+1}) + \left(k_{t+1}^{\theta_{t+1}}\beta_1 + \beta_2\right)(a+\delta_t)\right)\hat{x}_t - e_t - \left(i_t - i_t^f\right)\right] \\ &= \gamma b_t^* Var_t^{\theta_t}(J_{t+1})\end{aligned}$$

In equilibrium, $\beta_2^* = -1$, and

$$\left(\left(1-k_{t+1}^{\theta_{t+1}}\right)\beta_1^*(a+\delta_{t+1}) + \left(k_{t+1}^{\theta_{t+1}}\beta_1^* + \beta_2^*\right)(a+\delta_t)\right) = \beta_1^*$$

$$\Rightarrow \beta_1^* = \frac{a+\delta_t}{1-\left[\left(1-k_{t+1}^{\theta_{t+1}}\right)(a+\delta_{t+1}) + k_{t+1}^{\theta_{t+1}}(a+\delta_t)\right]}$$

Since $\beta_2^* = -1$ in equilibrium, the first order condition for δ_t becomes

$$
\begin{aligned}
0 &= \frac{\partial \Gamma}{\partial \delta_t} = -\left[\gamma b_t^* Var_t^{\theta_t}\left(J_{t+1}\right) + \frac{\left(\gamma b_t\right)^2}{2}\left(k_{t+1}^{\theta_{t+1}}\beta_1^* + \beta_2^*\right)^2\left(2\left(a+\delta_t\right)k_t + \left(a+\delta_t\right)^2\frac{dk_t}{d\delta_t}\right)\right] \times \\
& \exp\left(-\gamma b_t E_t^{\theta_t}\left(J_{t+1}\right) + \frac{\left(\gamma b_t\right)^2}{2} Var_t^{\theta_t}\left(J_{t+1}\right)\right) + \lambda \frac{E^{\theta_t} x_t^2}{\sigma_w^2}\delta_t
\end{aligned}
$$

This condition implies that $\delta_t^* > 0$. Since the gain k_t is an increasing function of δ_t, we conclude that $k_t^{\theta_t} > k^{\theta'}$.

Auxiliary Results

1. **Derivation of Equation (3.5).** Let $Z_{t+1} = \left[\exp(i_t) - \frac{E_{t+1}}{E_t}\exp(i_t^f)\right] b_t$ and $e_t := \log(E_t)$. The Taylor expansion around zero is given by

$$
\begin{aligned}
Z_{t+1} &= b_t\left[\exp(i_t) - \exp(e_{t+1} - e_t + i_t^f)\right] \\
&= b_t\left[(1 + i_t + o_2(2)) - (1 + e_{t+1} - e_t + i_t^f + o_1(2))\right] \\
&= b_t\left[-(e_{t+1} - e_t) + i_t - i_t^f + o\left(2\right)\right],
\end{aligned}
$$

where $o(2) = o_1(2) + o_2(2)$ and

$$
\begin{aligned}
o_1(2) &= -\frac{1}{2}\exp\left(\xi_1\right)\left(e_{t+1} - e_t + i_t^f\right)^2, \; o_2(2) = \frac{1}{2}\exp\left(\xi_2\right) i_t^2, \\
\xi_1 &\in \left(0, e_{t+1} - e_t + i_t^f\right), \; \xi_2 \in \left(0, i_t\right)
\end{aligned}
$$

Clearly, $\lim_{x_i \to 0}\frac{o_i(x_i)}{x_i} = 0$, for $i = 1, 2$ where $x_1 = e_{t+1} - e_t + i_t^f$ and $x_2 = i_t$. Thus the terms $o_1(2)$ and $o_2(2)$ are approximately zero if $e_{t+1} - e_t + i_t^f$ and i_t are small.

2. To show that if we let γ go to zero, the primitive utility function becomes risk neutral consider the following monotonic transformation of the primitive utility function: $-\frac{1}{\gamma}E^{\theta'}\left(\exp\left(-\gamma W_{t+1}\right) - 1\right).$

Lemma 8.3. *$lim_{\gamma \to 0} - \frac{1}{\gamma}E^{\theta}\left(\exp\left(-\gamma W_{t+1}\right) - 1\right) = E^{\theta}(W_{t+1})$.*

Proof. Applying L'Hopital's rule, we have

$$
\begin{aligned}
& \lim_{\gamma \to 0} E^{\theta}\left[-\frac{1}{\gamma}\left(\exp\left(-\gamma W_{t+1}\right) - 1\right) + \lambda \cdot \Re(\theta || \theta')\right] \\
=\; & E^{\theta}\left[\lim_{\gamma \to 0} W_{t+1}\exp\left(-\gamma W_{t+1}\right) + \lambda \cdot \Re(\theta || \theta')\right] \\
=\; & E^{\theta}\left[W_{t+1} + \lambda \cdot \Re(\theta || \theta')\right].
\end{aligned}
$$

3. We have defined the utility function in terms of the log excess return $W_{t+1} = b_t\left[(i_t - i_t^f) - (e_{t+1} - e_t)\right]$. The following Lemma shows that for a given domestic interest rate i_t, $u_1 = E_t^\theta\left[-\exp\left(-\gamma W_{t+1}\right)\right]$ is a monotonic transformation of $u_2 = E_t^\theta\left[-\exp\left(-\gamma b_t\left(\exp(i_t) - \frac{E_{t+1}}{E_t}\exp(i_t^f)\right)\right)\right]$.

Lemma 8.4. *For a fixed i_t, $u_1 = E^\theta\left\{-\exp\left[-\gamma b_t\left(\exp\left(i_t\right) - \frac{E_{t+1}}{E_t}\exp\left(i_t^f\right)\right)\right]\right\}$ is a monotonic transformation of $u_2 = E^\theta\left\{-\exp\left[-\gamma b_t\left(i_t - i_t^f - e_{t+1} + e_t\right)\right]\right\}$.*

Proof. Suppose that under utility function u_1

$$\left(b_{t,1}, E_{t+1,1}, E_{t,1}, i_{t,1}^f\right) \succcurlyeq \left(b_{t,2}, E_{t+1,2}, E_{t,2}, i_{t,2}^f\right)$$

$\Leftrightarrow$

$$\begin{aligned} u_1 &= E^\theta\left\{-\exp\left[-\gamma b_t\left(\exp\left(i_t\right) - \frac{E_{t+1,1}}{E_{t,1}}\exp\left(i_{t,1}^f\right)\right)\right]\right\} \\ &\geq u_2 = E^\theta\left\{-\exp\left[-\gamma b_t\left(\exp\left(i_t\right) - \frac{E_{t+1,2}}{E_{t,2}}\exp\left(i_{t,2}^f\right)\right)\right]\right\} \end{aligned}$$

for any measure θ.

$\Leftrightarrow$

$$\exp\left(i_t\right) - \frac{E_{t+1,1}}{E_{t,1}}\exp\left(i_{t,1}^f\right) \geq \exp\left(i_t\right) - \frac{E_{t+1,2}}{E_{t,2}}\exp\left(i_{t,2}^f\right)$$

$\Leftrightarrow$ dividing both sides by $\exp\left(i_t\right)$

$$1 - \frac{E_{t+1,1}}{E_{t,1}}\exp\left(i_{t,1}^f - i_t\right) \geq 1 - \frac{E_{t+1,2}}{E_{t,2}}\exp\left(i_{t,2}^f - i_t\right)$$

$\Leftrightarrow$

$$\exp\left(i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1}\right) \geq \exp\left(i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2}\right)$$

$\Leftrightarrow$

$$i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1} \geq i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2}$$

$\Leftrightarrow$

$$\begin{aligned} &-\exp\left[-\gamma b_t\left(i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1}\right)\right] \\ \geq\ &-\exp\left[-\gamma b_t\left(i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2}\right)\right] \end{aligned}$$

$$\begin{aligned} \tilde{u}_1 &\equiv E^\theta\left\{-\exp\left[-\gamma b_t\left(i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1}\right)\right]\right\} \\ &\geq \tilde{u}_2 \equiv E^\theta\left\{-\exp\left[-\gamma b_t\left(i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2}\right)\right]\right\} \end{aligned}$$

Therefore, $\left(E_{t+1,1}, E_{t,1}, i_{t,1}^f\right) \succcurlyeq \left(E_{t+1,2}, E_{t,2}, i_{t,2}^f\right)$.