Math 227 Fall 2008 Midterm Exam Answers

1. Evaluate $\int \frac{\sqrt{arctan\, x}}{1+x^2} dx$

Use substitution $u = arctan(x),\ du = dx/(1+x^2)$. So
$\int \frac{\sqrt{arctan\, x}}{1+x^2} dx = \int \sqrt{u} du = \frac{1}{3} u^{3/2} + c = \frac{1}{3}(arctan\, x)^{3/2} + c$

2. Evaluate $\int (ln\, x)^2\, dx$

By parts: $u = ln\, x,\ dv = \frac{1}{x} dx$, so $du = \frac{1}{x} dx$ and $v = x(ln\, x - 1)$ (see note below). Therefore
$\int (ln\, x)^2\, dx = x(ln\, x)(ln\, x - 1) - \int (ln\, x - 1)\, dx$
$= x(ln\, x)(ln\, x - 1) - x(ln\, x - 1) + x + c$
$= x((ln\, x)^2 - 2ln\, x + 2) + c$

Note: $\int 1.ln\, x\, dx = xln\, x - \int \frac{x}{x}\, dx = x(ln\, x - 1)$ by parts.

3. Evaluate $\int sec^4\, \theta\, d\theta$

$\int sec^4\, \theta\, d\theta = \int (sec^2\, \theta)(sec^2\, \theta)\, d\theta = \int (1 + tan^2\, \theta)(sec^2\, \theta)\, d\theta$
$= \int (1 + u^2)\, du = u + \frac{1}{3}u^3 + c = tan\, \theta + \frac{1}{3} tan^3\, \theta + c$
with $u = tan\, \theta$.

4. Evaluate $\int \frac{x^2}{(1+x)^3} dx$

By partial fractions
$\frac{x^2}{(1+x)^3} = \frac{A}{(1+x)^3} + \frac{B}{(1+x)^2} + \frac{C}{(1+x)} = \frac{(A+B+C)+(B+2C)x+Cx^2}{(1+x)^3}$
So, $A = 1, B = -2, C = 1$, and
$\int \frac{x^2}{(1+x)^3} dx = \int \frac{1}{(1+x)^3} dx - 2\int \frac{1}{(1+x)^2} dx + \int \frac{1}{(1+x)} dx$
$= -\frac{1}{2(1+x)^2} + \frac{2}{(1+x)} + ln\, |1 + x| + c$

5. Evaluate $\int_0^1 \frac{ln\, x}{\sqrt{x}}\, dx$

By parts $\int \frac{ln\, x}{\sqrt{x}}\, dx = 2x^{1/2} ln\, x - 2\int \frac{x^{1/2}}{x}\, dx = 2x^{1/2} ln\, x - 4x^{1/2} + c$

with $u = \ln x,\ dv = x^{-1/2}$ so that $du = \frac{dx}{x}$ and $v = 2x^{1/2}$. So
$\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx = \left(2x^{1/2}\ln x - 4x^{1/2}\right)\big|_{0^+}^{1} = -4 - 2\lim_{x\to 0^+} x^{1/2}\ln x = -4$
since $\lim_{x\to 0^+} x^{1/2}\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+} \frac{1/x}{-(1/2)x^{-3/2}} = -2\lim_{x\to 0^+} x^{1/2} = 0$
by l'Hopital. So, the improper integral is convergent and has value -4.

6. Sketch the region $\mathcal{R}$ bounded by $x = 0,\ x = 1,\ y = 0,\ y = e^{-x}$ and find the volume obtained by revolving $\mathcal{R}$ about the y-axis.

By the shell method, the volume is
$V = 2\pi \int_0^1 x\, e^{-x}\, dx = -2\pi\,(x+1)e^{-x}\big|_0^1 = 2\pi\left(1 - \frac{2}{e}\right)$
where $\int x\, e^{-x}\, dx = -x\, e^{-x} + \int e^{-x}\, dx = -(x+1)e^{-x}$ by parts.