Random Matrices

Suppose we pick a matrix at random in a given set. More specifically, suppose we are interested in hermitian matrices H (such that $H^* = H$) and that we wish to choose its coefficients randomly and independently, what is the probability that we hit one matrix rather than another? More precisely, what is the probability that we hit such and such eigenvalues for that matrix. These are the questions we now take up. We will restrict our attention to 2×2 hermitian matrices to get a sense of the issues. But most of the theory can be developed with $n \times n$ matrices for the hermitian, the symmetric, and other cases.

Let $H = \begin{pmatrix} x_{11} & x_{12} + iy_{12} \\ x_{12} - iy_{12} & x_{22} \end{pmatrix}$, with x_{11}, x_{12}, y_{12}, and x_{22} in **R**, be an arbitrary hermitian matrix. If the probabilities of choosing the entries are all independent, the probability density for any such matrix should be given by

$$P(H) = f_{11}(x_{11}) f_{12}(x_{12}) g_{12}(y_{12}) f_{22}(x_{22}) \qquad (1)$$

with *density* functions f_{jk} and $g_{12} > 0$ and a normalization condition $\int\int\int\int P(H) dH = 1$.

The first question is: "what kind of functions f_{jk} and g_{12} are possible?" The answer depends on additional conditions on P(H). For instance, we know that a hermitian matrix is unitarily equivalent to a real diagonal matrix: $U^*HU = \Lambda$ with $U^*U = I$ and Λ real diagonal. Since U corresponds to an orthonormal change of basis, H and Λ represent the same linear map and their probabilities should be the same. More generally, if H' is hermitian and U is unitary, $H = U^*H'U$ is also hermitian and represents the same linear map in another orthonormal basis. This similarity tranformation shouldn't change the probability of occurence since the measure of each element of n-dimensional space is left unchanged by an orthonormal change of basis. So, it is reasonable to assume that $P(U^*H'U) = P(H')$ for *any* unitary matrix U. Let us investigate the consequences of this assumption.

Consider the following (by no means most general) $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. It is clearly unitary (it is a rotation of angle $-\theta$), and let $H = U^*H'U$. By our assumption, a change in θ does not affect the probability of H. So, we can write from (1)

$$\frac{1}{P}\frac{dP}{d\theta} = \left(\frac{f_{11}'}{f_{11}}\frac{dx_{11}}{d\theta} + \frac{f_{12}'}{f_{12}}\frac{dx_{12}}{d\theta} + \frac{g_{12}'}{g_{12}}\frac{dy_{12}}{d\theta} + \frac{f_{22}'}{f_{22}}\frac{dx_{22}}{d\theta} \right) = 0 \qquad (2)$$

Now, let us calculate the $\frac{dx_{kl}}{d\theta}$ by differentiating $H = U^*H'U$ (and using $UH = H'U$ and $U^*H' = HU^*$)

$$\frac{dH}{d\theta} = \frac{dU^*}{d\theta}H'U + U^*H'\frac{dU}{d\theta} = \left(\frac{dU^*}{d\theta}U\right)H + H\left(U^*\frac{dU}{d\theta}\right) \tag{3}$$

or, if we let $A = U^*\frac{dU}{d\theta}$, $\frac{dH}{d\theta} = A^*H + HA$. But one finds easily:

$$A = U^*\frac{dU}{d\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So, one finds by direct calculation that

$$\frac{dH}{d\theta} = A^*H + HA = \begin{pmatrix} -2x_{12} & x_{11} - x_{22} \\ x_{11} - x_{22} & 2x_{12} \end{pmatrix} = \begin{pmatrix} \frac{dx_{11}}{d\theta} & \frac{dx_{12}}{d\theta} + i\frac{dy_{12}}{d\theta} \\ \frac{dx_{12}}{d\theta} - i\frac{dy_{12}}{d\theta} & \frac{dx_{22}}{d\theta} \end{pmatrix}$$

We can replace in (2) to find (note that $\frac{dy_{12}}{d\theta} = 0$ here)

$$2x_{12}\left(-\frac{f_{11}'}{f_{11}} + \frac{f_{22}'}{f_{22}}\right) + (x_{11} - x_{22})\frac{f_{12}'}{f_{12}} = 0$$

or

$$\frac{1}{2x_{12}}\frac{f_{12}'}{f_{12}} = \frac{1}{x_{11}-x_{22}}\left(\frac{f_{11}'}{f_{11}} - \frac{f_{22}'}{f_{22}}\right) \tag{4}$$

Now, the left-hand side of (4) depends only on x_{12} while the right-hand side depends only on x_{11} and x_{22}. So, the two sides of (4) equal some constant (say $-\alpha$). It follows that

$$\frac{f_{12}'}{f_{12}} = -2\alpha x_{12} \qquad \text{or} \qquad f_{12}(x_{12}) = c_{12}e^{-\alpha x_{12}^2} \tag{5}$$

Similarly, the right-hand side of (4) yields

$$\alpha x_{11} + \frac{f_{11}'}{f_{11}} = \alpha x_{22} + \frac{f_{22}'}{f_{22}} = \beta \tag{6}$$

which yields

$$f_{11}(x_{11}) = c_{11}e^{-\frac{1}{2}\alpha x_{11}^2 + \beta x_{11}} \qquad \text{and} \qquad f_{22}(x_{22}) = c_{22}e^{-\frac{1}{2}\alpha x_{22}^2 + \beta x_{22}}$$

By this investigation with the given unitary U, we have identified f_{11}, f_{12}, and f_{22} but *not* g_{12}. It is not too difficult to obtain $g_{12}(y_{12}) = d_{12}e^{-\alpha y_{12}^2}$ from a similar argument (see homework). Putting all results together, we find

$$P(H) = ce^{-\frac{1}{2}\alpha(x_{11}^2 + 2x_{12} + 2y_{12} + x_{22}^2) + \beta(x_{11} + x_{22})} \tag{7}$$

Of course, for this to be a true density function, it is necessary that $\alpha > 0$. One amazing fact (verify it) that can be generalized to the $n \times n$ case is that

$$\mathrm{P(H)} = \mathrm{c}e^{-\frac{1}{2}\alpha \mathrm{trace(H^2)} + \beta \mathrm{trace(H)}} \quad (8)$$

In all this, the constants α and β appeared from integration of differential equations and are arbitrary (except for the sign of α). So, additional assumptions will be needed to discuss probability distributions of hermitian matrices. But they are necessarily restricted to forms such as (8) as long as the two basic assumptions (independence of probabilities of the entries, and invariance under unitary transformation) are met. One typical case that we will focus on (next time) is

$$\mathrm{P(H)} = \mathrm{c}e^{-\mathrm{trace(H^2)}} \quad (9)$$

Homework

1. Check directly that P(H) given by (8) is invariant under unitary transformation. Hint: calculate $\mathrm{P(U^*H'U)}$.

2. Let $\mathrm{U} = \begin{pmatrix} \mathrm{a} & \mathrm{b}e^{i\varphi} \\ \mathrm{c}e^{i\psi} & \mathrm{d} \end{pmatrix}$ with a, b, c, d real positive and φ, $\psi \in [0,2\pi)$. Write conditions on a, b, c, d, φ, and ψ so that U is unitary.

3. Let $\mathrm{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{i\varphi} \\ -e^{-i\varphi} & 1 \end{pmatrix}$. Calculate $\frac{\mathrm{dU}}{\mathrm{d}\varphi}\big|_{\varphi=0}$ and $\mathrm{U}^*\big|_{\varphi=0}$. Then let $\mathrm{A} = \left(\mathrm{U}^* \frac{\mathrm{dU}}{\mathrm{d}\varphi}\right)\big|_{\varphi=0}$ and calculate $\frac{\mathrm{dH}}{\mathrm{d}\varphi} = \mathrm{A^*H + HA}$. Conclude that

$$\frac{1}{2\mathrm{y}_{12}}\frac{g_{12}'}{g_{12}} = \frac{1}{(\mathrm{x}_{11} - \mathrm{x}_{22} + 2\mathrm{x}_{12})}\left(\frac{f_{11}'}{f_{11}} - \frac{f_{22}'}{f_{22}} + \frac{f_{12}'}{f_{12}}\right) = -\alpha$$

and therefore that $g_{12}(\mathrm{y}_{12}) = \mathrm{d}_{12}e^{-\alpha \mathrm{y}_{12}^2}$.