Linear Algebra (continued)

We now investigate the eigenvectors associated to the eigenvalues found last time. First, consider $\lambda_1 = 2$. Let $v_1^T = (v_{11}, v_{21})$ be an associated eigenvector (any multiple of an eigenvector is still an eigenvector). It must satisfy $Av_1 = 2v_1$ or, row by row:

$$v_{11} + 2v_{21} = 2v_{11}$$

$$-v_{11} + 4v_{21} = 2v_{21}$$

which are *both* equivalent to $v_{11} = 2v_{21}$. So, we may choose for v_1 any (non-zero) components that satisfy this relation, for instance $v_1^T = (2,1)$. Similarly, one finds for $\lambda_2 = 3$ the single condition $v_{21} = v_{22}$ and a possible eigenvector $v_2^T = (1,1)$. Now, we can return to the initial question that motivated our quest: "could we possibly transform A into a diagonal matrix by a similarity transformation?" To see why we can for this choice of A, we only need to rewrite our defining condition for eigenvalues and eigenvectors $Av = \lambda v$ for the each of the two above possibilities using a single interesting trick: if we set $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ then the multiplication of any matrix B by Λ *on the right* has the effect of multiplying the first row of B by λ_1 and the second row of B by λ_2 (check it and generalize it to the $n \times n$ case). Let us try this trick for the matrix B whose columns are v_1 and v_2. We find (check it!)

$$AB = B\Lambda \tag{1}$$

Now, it turns out that B is nonsingular (we will see a general principle for this). So, we can multiply both sides of (1) by B^{-1} to find

$$B^{-1}AB = \Lambda \tag{2}$$

which means that A is similar to the diagonal matrix Λ (or that A is diagonalizable).

Let us now consider the case $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The characteristic polynomial reads $p(\lambda) = (1-\lambda)^2 + 1$ with roots $\lambda = 1 \pm i$. We can no longer hope to find *real* eigenvectors since Av would be real while λv would not! Indeed, we cannot even hope to diagonalize the matrix A with *real* numbers since that would mean $B^{-1}AB = \Lambda$ or $AB = B\Lambda$ with B and Λ real and Λ diagonal. But this would mean that the diagonal elements of Λ would be eigenvalues and we cannot have any others than those already found since the characteristic polynomial of a 2×2 matrix is quadratic. However, we can diagonalize A with complex B and Λ.

A third interesting case is given by $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The characteristic polynomial now reads $p(\lambda) = (1 - \lambda)^2$ with *double* root $\lambda = 1$. The trouble here is that the equation $Av = v$ only has a one-dimensional family of solutions, those proportional to $v^T = (0,1)$. So, the trick $AB = B\Lambda$ requires that the two columns in B be proportional. Thus, B would be singular and one cannot diagonalize A in **R** or **C**.

Homework

1. Find B^{-1} for B in (1) and verify (2) directly.

2. Diagonalize $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ with complex B and Λ.