Answers to Chapter 5 Homework

Problem 1.

Let b denote Green's belief to be at node G1. There are three cases:

1) If $b < b^* = \frac{1}{4}$ then the expected payoffs E^G_{left} and E^G_{right} for Green of choices Left and Right respectively satisfy

$$E^B_{left} = 1 + 2b < E^G_{right} = 2 - 2b$$

So, Green chooses Right with resulting expected payoff $E^R_{\mathrm{G2}} = -1$ for Red at node G2. It follows that Red chooses Right at node R with expected payoff $E^B_{\mathrm{R}} = 2$ for Blue at node R. Since Blue expects $E^B_{\mathrm{G1}} = -1$ at node G1 from Green's choice, he rationally chooses Right at node B. $\{$G1,G2$\}$ is thus off the equilibrium path and this allows any arbitrary belief $b < \frac{1}{4}$. This yields a continuum of PBEs $\{$Right, Right, Right, $b < \frac{1}{4}\}$.

2) If $b > \frac{1}{4}$ then the expected payoffs for Green satisfy $E^G_{left} > E^G_{right}$, Green chooses Left with resulting expected payoff $E^R_{\mathrm{G2}} = 1$ for Red at node G2. Red thus chooses Down with expected payoff $E^B_{\mathrm{R}} = E^B_{\mathrm{G2}} = 1$ for Blue at node R. Since $E^B_{\mathrm{G1}} = 3$, Blue chooses down at node B. The (updated) belief is therefore $b = 1$, consistent with $b > \frac{1}{4}$ and we have a single PBE $\{$Down, Down, Left, $b = 1\}$.

3) If $b = \frac{1}{4}$ then Green is indifferent between Left and Right and can choose the latter with probability r. This yields $E^R_{\mathrm{G2}} = 1 - 2r$. There are three subcases:
a) $r > \frac{1}{2}$ implies $E^R_{\mathrm{G2}} < 0$, Red chooses Right at node R and $E^B_{\mathrm{G1}} < 1 < E^B_{\mathrm{R}} = 2$. So, Blue chooses Right, $\{$G1,G2$\}$ is off the equilibrium path and the arbitrary belief $b = \frac{1}{4}$ provides a PBE.
b) $r < \frac{1}{2}$ implies $E^R_{\mathrm{G2}} > 0$, so Red chooses Down at node R and

$$E^B_{\mathrm{G1}} = 3 - 4r > E^B_{\mathrm{R}} = E^B_{\mathrm{G2}} = 1 - 2r$$

so that Blue chooses Down. But updated belief is $b = 1$ at node G1, a contradiction.
c) $r = \frac{1}{2}$ implies that Red can chooses Right with probability q. This results in $E^B_{\mathrm{R}} = 2q$ while $E^B_{\mathrm{G1}} = 1$. If $q > \frac{1}{2}$ Blue chooses Right and updated belief $b = 0$ is incompatible with $b = \frac{1}{4}$. If $q < \frac{1}{2}$ Blue chooses Down and updated belief $b = 1$ is incompatible with $b = \frac{1}{4}$.
If $q = \frac{1}{2}$ Blue can choose Down with probability p. Updated belief b must then satisfy $b = \frac{p}{p+q(1-p)} = \frac{1}{4}$ so that $p = \frac{1}{7}$. This yields a single PBE.

In summary, there are two continuums of PBEs and two isolated ones as follows:
- $\{$Right, Right, Right, $b < \frac{1}{4}\}$;
- $\{$Right, Right, $r > \frac{1}{2}, b = \frac{1}{4}\}$;
- $\{$Down, Down, Left, $b = 1\}$;
- $\{p = \frac{1}{7}, q = \frac{1}{2}, r = \frac{1}{2}, b = \frac{1}{4}\}$.

Problem 2.

Let p denote the initial belief that defender is weak. Let b denote challenger's belief he is at node C3. Challenger's resulting expected payoffs E^B_{back} for backdown and E^B_{esc} for escalate are

$$E^B_{back} = -2b - (1-b) = -(1+b)$$

and $$E^B_{esc} = 2b - 2(1-b) = 4b - 2$$

We have three cases:

1) If $b < b^* = \frac{1}{5}$ then $E^B_{back} > E^B_{esc}$ and the challenger chooses backdown. The resulting defender's expected payoffs are best for resist that is rationally chosen at both nodes D1 and D2. The resulting expected payoffs for challenger at D1 and D2 are less than those of staying. Therefore the challenger always chooses stay, the information set $\{$C3,C4$\}$ is off the equilibrium path, and the arbitrary beliefs $b < b^*$ can be adopted at C3. This always yields a set of pooling PBEs with arbitrary initial beliefs p on $\{$C1,C2$\}$.

2) If $b > \frac{1}{5}$ then $E^B_{back} < E^B_{esc}$ and the challenger chooses escalate. The defender chooses submit at D1 and resist at D2. With the expected payoffs for challenger $E^B_{\text{D1}} = 1$ at D1 and $E^B_{\text{D2}} = -2$ at D2, the expected payoff of challenge is $E^B_{chal} = 3p - 2$, while the expected payoff of stay is $E^B_{chal} = 0$. $\{$C3,C4$\}$ can be off the equilibrium path if $3p - 2 \leq 0$ or $p \leq \frac{2}{3}$. In that case, there is a (whole range of) separating PBE where the challenger chooses stay and belief $b < \frac{1}{5}$ is chosen arbitrarily off the equilibrium path. If $\{$C3,C4$\}$ was *on* the equilibrium path (challenger chooses challenge at $\{$C1,C2$\}$ when $p > \frac{2}{3}$) the updated beliefs would be $b = 0$, a contradiction. So, a PBE exists in this case only if $p \leq \frac{2}{3}$.

3) If $b = b^*$ then $E^B_{back} = E^B_{esc}$. The challenger may use escalate with probability x. The expected payoffs for defender at C3 and C4 are

$$E^R_{\text{C3}} = -2x + 2(1-x) = 2 - 4x$$

and $$E^R_{\text{C4}} = 2x + (1-x) = 1 + x$$

Clearly, $E^R_{\text{C4}} > 1$ (defender's payoff of submit at D2). So, defender always resist at D2. However, there are three subcases for E^R_{C3}:

a) $E^R_{\text{C3}} > -1$ corresponds to $x < \frac{3}{4}$ and defender chooses resist at D1. The resulting expected payoffs of challenge is

$$E^B_{chal} = p(4x-2) - (1+x)(1-p) = (5p-1)x - (1+p)$$

For $p \leq \frac{1}{5}$ this is clearly negative and challenger will stay. Any probability $x < \frac{3}{4}$ together with off-equilibrium-path belief $b = \frac{1}{5}$ then provides a PBE. For $p > \frac{1}{5}$ and $x \leq \frac{1+p}{5p-1}$ we have a PBE, again with off-equilibrium-path belief $b = \frac{1}{5}$. The case where $E^B_{chal} > 0$ so that challenger chooses challenge is compatible with updated beliefs $b = \frac{1}{5}$ only if $p = \frac{1}{5}$. This also allows the whole range $x > \frac{3}{4}$. In sum, there is always a pooling equilibrium such that challenger stays, defender always resists and challenger escalates with probability $x < \frac{3}{4}$ and such that $x \leq \frac{1+p}{5p-1}$ when $p > \frac{1}{5}$. This subcase agrees in the limit with case 1 above.

b) $E^R_{\text{C3}} < -1$ (defender's payoff of submit at D1) corresponds to $x > \frac{3}{4}$ and defender chooses submit at D1. The expected payoff of challenge is (since $E^B_{\text{D2}} = -(1+x)$)

$$E^B_{chal} = p - (1+x)(1-p)$$

PBEs do exist if $E^B_{chal} \leq 0$, allowing stay to be optimal, when $x \geq \frac{2p-1}{1-p}$. This is always compatible with $x > \frac{3}{4}$ but requires $p \leq \frac{2}{3}$ (since $x \leq 1$). The belief $b = \frac{1}{5}$ can then be arbitrarily chosen since {C3,C4} is off the equilibrium path. However, $E^B_{chal} > 0$ yields updated belief $b = 0$, a contradiction. This subcase agrees in the limit with case 2 above.

c) $E^R_{\text{C3}} = -1$ corresponds to $x = \frac{1}{3}$ and the defender can choose resist with some probability r at D1. The cases $r \in \{0,1\}$ appear in the limit in the above two subcases. So, we focus on a true probability $r \in (0,1)$. The value of x here yields (for any of r):

$$E^B_{chal} = p - \tfrac{7}{4}(1-p) = \tfrac{11p-7}{4}$$

If $p \leq \frac{7}{11}$ challenger chooses stay, {C3,C4} is off the equilibrium path and beliefs $b = \frac{1}{5}$ can be arbitrarily picked together with an arbitrary r. If $p > \frac{7}{11}$ one obtains by Bayesian updating

$$b = \frac{pr}{pr+(1-p)} = \frac{1}{5} \qquad \text{for} \qquad r = \frac{1-p}{4p} \qquad \text{(always a probability if } p > \tfrac{7}{11}\text{)}$$

So, $b = \frac{1}{5}$ and $x = \frac{1}{3}$ always provides a semi-separating PBE whether {C3,C4} is off or on the equilibrium path. In the latter case $r = \frac{1-p}{4p}$ if $p > \frac{7}{11}$.

In summary, the following PBEs exist under the given conditions:

- (pooling): The challenger chooses stay and backdown, the defender always resist and the belief is $b < \frac{1}{5}$ at C3 for any initial beliefs p at C1. There are also limit cases with $b = \frac{1}{5}$ and low enough probability of escalate.

- (separating): The challenger chooses stay and escalate, the defender chooses submit at D1 and resist at D2, the initial beliefs are $p \leq \frac{2}{3}$ at C1 and the belief is $b > \frac{1}{5}$ at C3. There are also limit cases with $b = \frac{1}{5}$ and high enough probability of escalate.

• (semi-separating): The challenger escalates with probability $x = \frac{1}{3}$ and the defender resists with some probability r at D1. With initial beliefs $p \leq \frac{7}{11}$ challenger stays and r is arbitrary. With $p > \frac{7}{11}$ challenger challenges and defender resist with $r = \frac{1-p}{4p}$ at D1.