Appendix One: Nash's Theorem*

A (finite) game in normal form $\mathcal{G}$ is defined by a list $\{1, .., i, .., n\}$ of n players, a set of pure strategies $\mathcal{A}_i = \{\alpha_1^i, .., \alpha_{k(\text{i})}^i\}$ for each player, and a payoff function $\pi_i : \mathcal{A} = \underset{i=1}{\overset{n}{\times}} \mathcal{A}_i \to \Re$ for each player which associates the payoff $\pi_i(\alpha^1, .., \alpha^n)$ for player i, to any pure strategy n-tuple $\{\alpha^1, .., \alpha^n\}$.

A mixed strategy x^i for player i is a probability distribution $\{x_1^i, .., x_{k(i)}^i\}$ over i's pure strategies. A mixed strategy n-tuple $x = \{x^1, .., x^n\}$ is a list of n probability distributions (one for each player over that player's pure strategy set). The set $\mathcal{B}$ of all n-tuples x is clearly a compact (closed and bounded) and convex subset of the Euclidean space $\Re^m$ with $m = \sum_{i=1}^{n} k(i)$.

The expected payoff resulting from a mixed strategy n-tuple x is the multilinear extension of π_i (i.e., it is linear in each x^i). It will be convenient to denote by $\pi_i(x^i, x^{-i})$ the expected payoff when player i uses x^i and all other players together use the $(n-1)$-tuple x^{-i}.

We will prove:

Nash's Theorem: Any (finite) game in normal form has a Nash equilibrium.

We will need:

Lemma: $x \in \mathcal{B}$ is a Nash equilibrium if and only if for any i and any $\alpha \in \mathcal{A}_i$

$$\pi_i(\alpha, x^{-i}) \leq \pi_i(x^i, x^{-i}) \qquad (1)$$

Proof: Clearly, no pure strategy α is a better reply than x^i to x^{-i} for player i if (1) holds. Thus, for any mixed strategy $y^i = \{y_\alpha^i | \alpha \in \mathcal{A}_i\}$ and by linearity for each i:

$$\pi_i(y^i, x^{-i}) = \sum_{\alpha \in \mathcal{A}_i} y_\alpha^i \pi_i(\alpha, x^{-i}) \leq \left(\sum_{\alpha \in \mathcal{A}_i} y_\alpha^i\right) \pi_i(x^i, x^{-i}) \leq \pi_i(x^i, x^{-i}) \qquad (2)$$

and y^i is no better than x^i in response to x^{-i}. $Q.E.D.$

Proof of the theorem: For any pure strategy α, let $i(\alpha)$ be such that $\alpha \in \mathcal{A}_i$, and let:

$$u_\alpha(x) = max\{0, \pi_i(\alpha, x^{-i}) - \pi_i(x^i, x^{-i})\} \qquad (3)$$

for $i = i(\alpha)$. Clearly, u_α is continuous in x since π_i is. Further let $\mathcal{A}(\alpha) = \mathcal{A}_i$ for $i = i(\alpha)$ and:

$$y_\alpha = \frac{x_\alpha + u_\alpha}{1 + \sum_{\beta \in \mathcal{A}(\alpha)} u_\beta} \qquad (4)$$

where x_α is the probability of α in x. Evidently, $y_\alpha \geq 0$ and $\sum_{\alpha \in \mathcal{A}(\alpha)} y_\alpha = 1$. We can thus define y^i as a probability distribution of components y_α for $\alpha \in \mathcal{A}(\alpha)$, and let $y = \{y^1, .., y^n\}$ be the corresponding n-tuple of mixed strategies. Moreover, since y_α is clearly continuous in x, so is y as a function of x. Therefore, by Brouwer's theorem, the continuous map $\phi : \mathcal{B} \rightarrow \mathcal{B}$ defined by $y = \phi(x)$ must have a fixed point $z = \phi(z) \in \mathcal{B}$.

We finally verify that z is a Nash equilibrium. We first observe that for any i there must exist at least one $\alpha \in \mathcal{A}_i$ such that $z_\alpha > 0$ and $u_\alpha = 0$. Indeed, were it not the case, we would have $\pi_i(\alpha, z^{-i}) > \pi_i(z^i, z^{-i})$ for all $z_\alpha > 0$ and thus:

$$\pi_i(z^i, z^{-i}) = \sum_{\alpha \in \mathcal{A}_i} z_\alpha \pi_i(\alpha, z^{-i}) > \left(\sum_{\alpha \in \mathcal{A}_i} z_\alpha\right) \pi_i(z^i, z^{-i}) = \pi_i(z^i, z^{-i}) \tag{5}$$

a contradiction. But if $z_\alpha > 0$ and $u_\alpha = 0$ for some $\alpha \in \mathcal{A}_i$ then, for that α:

$$z_\alpha = \frac{z_\alpha}{1 + \sum_{\beta \in \mathcal{A}(\alpha)} u_\beta} \tag{6}$$

so that $\sum_{\beta \in \mathcal{A}(\alpha)} u_\beta = 0$ and $u_\beta = 0$ for *all* $\beta \in \mathcal{A}(\alpha)$. It follows that $\pi_i(\beta, z^{-i}) \leq \pi_i(z^i, z^{-i})$ for all $\beta \in \mathcal{A}_i$ and for all i. By the above lemma, z is a Nash equilibrium. $Q.E.D.$