

Math 370 Midterm #2 Sample

1. Show that any cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ (with $a \neq 0, b, c, d$ real) has at least one real root. Hint: Assuming $a > 0$, rewrite $p(x) = x^3 \cdot r(x)$ and argue that there exists $M > 0$ such that $r(x) \geq \frac{1}{2}a$ for $|x| \geq M$.

$r(x) = a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}$ clearly approaches a as $|x| \rightarrow \infty$ since $\frac{1}{x}$ approaches 0. So, there must exist $M > 0$ such that $r(x) \geq \frac{1}{2}a$ for $|x| \geq M$.

Now, $p(M) = M^3 r(M) \geq \frac{1}{2}aM^3 > 0$ and $p(-M) = -M^3 r(M) \leq -\frac{1}{2}aM^3 < 0$. Since the continuous function p changes sign on the interval $[-M, M]$ it must have a root by the intermediate value theorem.

2. Assume that the sequences $\{x_n\}$ and $\{y_n\}$ both converge to the same limit L . Define the sequence $\{z_n\}$ by:

$$z_n = \begin{cases} x_k & \text{if } n = 2k - 1 \\ y_k & \text{if } n = 2k \end{cases}$$

Using the definition of convergent sequences, prove that $\{z_n\}$ converges to L .

$\forall \epsilon > 0 \exists K_x \in \mathbb{N}$ such that $k \geq K_x \Rightarrow |x_k - L| < \epsilon$

and (for the same ϵ) $\exists K_y \in \mathbb{N}$ such that $k \geq K_y \Rightarrow |y_k - L| < \epsilon$

So, if I let $N = \max\{2K_x - 1, 2K_y\}$ then $n \geq N \Rightarrow \{|z_n - L| = |x_k - L| < \epsilon$ if $n = 2k - 1$ or $|z_n - L| = |y_k - L| < \epsilon$ if $n = 2k\}$.

3. Assume that $f : \mathcal{I} = [a, b] \rightarrow \mathbb{R}$ is continuous and that there exists a unique $c \in \mathcal{I}$ such that $f(x) = 0 \Leftrightarrow x = c$. Assume that $\{x_n\}$ is a sequence in \mathcal{I} such that $\lim_{n \rightarrow \infty} f(x_n) = 0$. Prove that $\{x_n\}$ converges to c using the following theorem: "if all convergent subsequences of a bounded sequence converge to a same limit c then the sequence converges to c ."¹

Since x_n is bounded consider an arbitrary convergent subsequence $\{x_{n_k}\}$ with limit x . Since $\{f(x_{n_k})\}$ is a subsequence of the convergent sequence $\{f(x_n)\}$ it has the same limit 0. By the sequential characterization theorem $f(x) = \lim_{k \rightarrow \infty} f(x_{n_k}) = 0$. By

assumption it follows that $x = c$. By the theorem, since all convergent subsequences $\{x_{n_k}\}$ have the same limit c , $\{x_n\}$ converges to c .

4. Assume that $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\forall x, y : g(x + y) = g(x) \cdot g(y)$. Show that if g is continuous at 0 then g is continuous everywhere in \mathbb{R} .

If g is continuous at 0 then (by definition) $\lim_{x \rightarrow 0} g(x) = g(0)$. But then:

$$\begin{aligned} \lim_{z \rightarrow y} g(z) &= \lim_{x \rightarrow 0} g(x + y) && \text{(since } z \rightarrow y \Leftrightarrow z = x + y \text{ and } x \rightarrow 0) \\ &= \lim_{x \rightarrow 0} (g(x) \cdot g(y)) = \left(\lim_{x \rightarrow 0} g(x)\right) \cdot g(y) && \text{(by limit theorems)} \\ &= g(0) \cdot g(y) = g(0 + y) = g(y) \end{aligned}$$

So, g is continuous at any y .

¹This theorem appears in other textbooks but not yours. The proof is by contradiction: if $\{x_n\}$ does not converge to c it has (by Bolzano-Weierstrass) a subsequence that converges but not to c .