

The Causal Interpretation of Quantum
Mechanics and The Singularity Problem in
Quantum Cosmology

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Abstract

We apply the causal interpretation of quantum mechanics to homogeneous quantum cosmology and show that the quantum theory is independent of any time-gauge choice and there is no issue of time. We exemplify this result by studying a particular minisuperspace model where the quantum potential driven by a prescribed quantum state prevents the formation of the classical singularity, independently on the choice of the lapse function. This means that the fast-slow-time gauge conjecture is irrelevant within the framework of the causal interpretation of quantum cosmology.

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1 Introduction

The singularity theorems [1] show that, under reasonable physical assumptions, the Universe has developed an initial singularity, and will develop future singularities in the form of black holes and, perhaps, of a big crunch. Until now, singularities are out of the scope of any physical theory. If we assume that a physical theory can describe the whole Universe at every instant, even at its possible moment of creation (which is the best attitude because it is the only way to seek the limits of physical science), then it is necessary that the ‘reasonable physical assumptions’ of the theorems be not valid under extreme situations of very high energy density and curvature. We may say that general relativity, and/or any other matter field theory, must be changed under these extreme conditions. One good point of view is to think that quantum gravitational effects become important, eliminating the singularities that should appear classically, similarly to what happens with the quantum atom. We should then construct a quantum theory of gravitation, apply it to cosmology, and see if it works. However, there is no established theory of quantum gravity. Furthermore, any quantum theory when applied to cosmology presents new profound conceptual problems. How can we apply the standard probabilistic Copenhagen interpretation to a single system? Where in a quantum Universe can we find a classical domain where we could construct our classical measuring apparatus to test and give sense to the quantum theory? Who are the observers of the whole Universe? This is not a problem of quantum gravity alone because there is no problem with the concept of an ensemble of black holes and a classical domain outside it. Finally, in quantum mechanics, time is not treated as an observable (hermitean operator) but as an external evolution parameter (c-number). In the quantum cosmology of a closed universe, there is no place for an external parameter. So, what happens with time? Which internal variable will give a sense of evolution of the quantum states?

In this paper we will close our attention to the interpretation and time

issues in order to study the singularity problem in quantum cosmology. The difficult technical problems coming from the quantization of the full gravitational field will be circumvented by taking advantage of minisuperspace models which restrict the gravitational and matter fields to be homogeneous. In these models, all but a finite number of degrees of freedom are frozen out alleviating considerably the technical problems.

In the framework of these minisuperspace models, a number of papers have been written showing how the issue of time is important for the singularity problem: different choices of time imply different quantum theories, some of them still presenting singularities, others not [2, 3]. The interpretation adopted is the conventional probabilistic one. Here, we will adopt a non-probabilistic interpretation to quantum cosmology which circumvents the measurement problem because it is an ontological interpretation of quantum mechanics: it is not necessary to have a measuring apparatus or a classical domain in order to recover physical reality; it is there “ab initio”. It is the causal interpretation of quantum mechanics [4, 5]. We will apply this interpretation to the minisuperspace models of homogeneous gravitational and matter fields mentioned above, and show that the question about the persistency of the singularities at the quantum level does not depend on the choice of time but only on the quantum state of the system. A particular example will be exhibited to bring home this fact.

This paper is organized as follows: in the next section we make a summary of the causal interpretation. In section 3, we apply this interpretation to quantum cosmology, and show that, for the minisuperspace models of homogeneous gravitational and matter fields, the quantum theory is independent on the choice of time. We also call attention to the fact that this result may no longer be valid for inhomogeneous fields. In section 4, we present a particular minisuperspace example where the classical singularities can be removed by a choice of the quantum state, and show that this result does not depend on the choice of time. We end with some comments and conclusions.

2 The causal interpretation of quantum mechanics

In this section, we will review the ontological interpretation of quantum mechanics, and apply it to quantum cosmology. Let us begin with the Schrödinger equation, in the coordinate representation, for a non-relativistic particle with the hamiltonian $H = p^2/2m + V(x)$:

$$i\hbar \frac{d\Psi(x, t)}{dt} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(x)\right] \Psi(x, t). \quad (1)$$

Writing $\Psi = R \exp(iS/\hbar)$, and substituting it into (1), we obtain the following equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = 0, \quad (2)$$

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(R^2 \frac{\nabla S}{m} \right) = 0. \quad (3)$$

The usual probabilistic interpretation takes equation (3) and understands it as a continuity equation for the probability density R^2 for finding the particle at position x and time t . All physical information about the system is contained in R^2 , and the total phase S of the wave function is completely irrelevant. In this interpretation, nothing is said about S and its evolution equation (2). However, examining equation (3), we can see that $\nabla S/m$ may be interpreted as a velocity field, suggesting the identification $p = \nabla S$. Hence, we can look to equation (2) as a Hamilton-Jacobi equation for the particle with the extra potential term $-\hbar^2 \nabla^2 R / 2mR$.

After this preliminary, let us introduce the ontological interpretation of quantum mechanics, which is based on the *two* equations (2) and (3), and not only in the last one as it is the Copenhagen interpretation:

i) A quantum system is composed of a particle *and* a field Ψ (obeying the Schrödinger equation (1)), each one having its own physical reality.

ii) The quantum particles follow trajectories $x(t)$, *independent on observations*. Hence, in this interpretation, we can talk about trajectories of

quantum particles, contrary to the Copenhagen interpretation where only positions at one instant of time have a physical meaning.

iii) The momentum of the particle is $p = \nabla S$.

iv) For a statistical ensemble of particles in the same quantum field Ψ , the probability density is $P = R^2$. Equation (3) guarantees the conservation of P .

Let us make some comments:

a) Equation (2) can now be interpreted as a Hamilton-Jacobi type equation for a particle submitted to an external potential which is the classical potential plus a new quantum potential

$$Q \equiv -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}. \quad (4)$$

Hence, the particle trajectory $x(t)$ satisfies the equation of motion

$$m \frac{d^2 x}{dt^2} = -\nabla V - \nabla Q. \quad (5)$$

b) Even in the regions where Ψ is very small, the quantum potential can be very high, as we can see from equation (4). It depends only on the form of Ψ , not on its absolute value. This fact brings home the non-local and contextual character of the quantum potential¹. This is very important because Bell's inequalities together with Aspect's experiments show that, in general, a quantum theory must be either non-local or non-ontological. As Bohm's interpretation is ontological, it must be non-local, as it is. The quantum potential is responsible for the quantum effects.

c) This interpretation can be applied to a single particle. In this case, equation (3) is just an equation to determine the function R , which forms the quantum potential acting on the particle via equation (5). The function R^2 does not need to be interpreted as a probability density and hence needs not be normalized. The interpretation of R^2 as a probability density is

¹This fact becomes evident when we generalize the causal interpretation to a many particle system.

appropriate only in the case mentioned in item (iv) above. The ontological interpretation is not, in essence, a probabilistic interpretation.

d) The classical limit is very simple: we only have to find the conditions for having $Q = 0$.

e) There is no need to have a classical domain because this interpretation is ontological. The question on why in a real measurement we do not see superpositions of the pointer apparatus is answered by noting that, in a measurement, the wave function is a superposition of non-overlapping wave functions [6]. The particle will enter in one region, and it will be influenced by the unique quantum potential obtained from the sole non-zero wave function defined on this region.

Of course this interpretation has still some flaws. It is difficult to accommodate it with the notion of spin, it works only in the coordinate representation [7], its generalization to quantum fields is not yet completely understood (see however [8]), just to mention some of them. Nevertheless, as it is an interpretation which does not require a classical domain, and which can be applied to a single system, we think it should be relevant to examine what it can say about quantum cosmology.

3 The application of the causal interpretation to quantum cosmology

The hamiltonian of General Relativity (GR) without matter is given by:

$$H_{GR} = \int d^3x (N\mathcal{H} + N_j\mathcal{H}^j), \quad (6)$$

where

$$\mathcal{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - h^{1/2}R^{(3)}, \quad (7)$$

$$\mathcal{H}^j = -2D_i\Pi^{ij}. \quad (8)$$

The momentum Π_{ij} canonically conjugated to the space metric h^{ij} of the spacelike hypersurfaces which foliate spacetime is

$$\Pi_{ij} = \frac{\delta L}{\delta(\partial_t h^{ij})} = -h^{1/2}(K_{ij} - h_{ij}K), \quad (9)$$

where

$$K_{ij} = -\frac{1}{2N}(\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_i), \quad (10)$$

and

$$G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}), \quad (11)$$

which is called the DeWitt metric. The quantity $R^{(3)}$ is the intrinsic curvature of the hypersurfaces and h is the determinant of h_{ij} . The lapse function N and the shift function N_j are the Lagrange multipliers of the super-hamiltonian constraint \mathcal{H} and the super-momentum constraint \mathcal{H}^j , respectively. They are present due to the invariance of GR under spacetime coordinate transformations. Their specifications fix the coordinates.

If we follow the Dirac quantization procedure, these constraints become conditions imposed on the possible states of the quantum system, yielding the following quantum equations:

$$D_j \frac{\delta \Psi(h^{ij})}{\delta h^{ij}} = 0 \quad (12)$$

$$(G^{ijkl} \frac{\delta}{\delta h^{ij}} \frac{\delta}{\delta h^{kl}} + h^{1/2} R^{(3)}) \Psi(h^{ij}) = 0 \quad (13)$$

(we have set $\hbar = 1$).

The first equation has a simple interpretation. It means that the value of the wave function does not change if the spacelike metric changes by a coordinate transformation.

The second one is the Wheeler-DeWitt equation, which should determine the evolution of the wave function. However, time has disappeared from it. There should exist one momentum which is canonically conjugate to

some intrinsic time in which the quantum dynamics takes place. In the time reparametrization invariant formulation of the quantum mechanics of a non-relativistic particle, this particular momentum is easily distinguishable from the others because it appears linearly in the quantum equation analogous to (13), while the others appear quadratically. However, in equation (13), there is no momentum which appears linearly; all of them appear quadratically. Hence, where is time? This is the famous issue of time. This fact makes people advocates another quantization scheme, the ADM approach, where time is chosen before quantization by a gauge fixing procedure. However, different choices of time lead to inequivalent quantum theories [2, 3] and there is no criterium to choose one of them.

Others say that the fact that it is not easy to find what should play the role of time in the Wheeler-DeWitt equation simply means that there is no time at all in quantum gravity [9, 10]. In fact, the good analogy with the time reparametrization invariant quantum mechanics of non-relativistic particles is via the Jacobi action:

$$S = \int d\tau \sqrt{F_E T}, \quad (14)$$

where $F_E \equiv E - V$ and $T = \frac{1}{2} \sum_{i=1}^n m_i \frac{dx^i}{d\tau} \frac{dx^i}{d\tau}$. This is the appropriate action when a closed conservative system is studied. The conserved energy is E , and V and T are the potential and kinetic energies of the system. This action yields Newton's equations of motion if a suitable choice of the parameter τ is made such that $T = F_E$. The hamiltonian can be calculated in the same way as before and it turns out to be proportional to the following constraint:

$$\frac{1}{2} \sum_{i=1}^n \frac{p^i p^i}{m_i} - F_E \approx 0. \quad (15)$$

Following the Dirac quantization scheme, this constraint yields the following quantum equation:

$$\frac{1}{2} \left(\sum_{i=1}^n \frac{\hat{p}^i \hat{p}^i}{m_i} + V \right) \Psi(x^i) = E \Psi(x^i), \quad (16)$$

which is the time independent Schrödinger equation. This is the correct analogous equation to the Wheeler-DeWitt equation (13) because it is also quadratic in all momenta. Consequently, we should consider the Wheeler-DeWitt equation as a time-independent Schrödinger equation with zero energy. This is consistent with the fact that a closed Universe has, by definition, a null total energy.

Using a non-ontological interpretation, we can understand this fact in another way. Space geometry is like position in ordinary particle mechanics while spacetime geometry is like a trajectory. Trajectories have no physical meaning in the quantum mechanics of particles following a non-epistemological interpretation. Instantaneous positions have. Analogously, spacetime has no physical meaning in quantum gravity, only space geometries have. Hence, time makes no sense at the Planck scale. Space is the most primitive concept [9, 10]. Therefore, it is quite natural that the Wheeler-DeWitt equation of closed spaces be time independent. It is a time independent Schrödinger equation for zero energy, as it should be!

However, if we apply the ontological interpretation to quantum cosmology, we should expect that the notion of a spacetime would have a meaning exactly like the notion of trajectories have in the causal interpretation of quantum mechanics of non-relativistic particles. Hence, we should expect that the notion of time would emerge naturally in this interpretation. Indeed, following the steps we made in order to describe the ontological interpretation in the beginning of this section, we substitute $\Psi = R \exp(iS/\hbar)$ into the Wheeler-DeWitt equation (13), yielding the two equations (for simplicity we stay in pure gravity):

$$\frac{1}{2}G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} - h^{1/2}R^{(3)}(h_{ij}) + h^{1/2}Q(h_{ij}) = 0, \quad (17)$$

$$G_{ijkl}\frac{\delta}{\delta h_{ij}}\left(R^2\frac{\delta S}{\delta h_{kl}}\right) = 0, \quad (18)$$

where the quantum potential is given by:

$$Q = -\frac{1}{R}G_{ijkl}\frac{\delta^2 R}{\delta h_{ij}\delta h_{kl}}. \quad (19)$$

As before, we postulate that $h^{ij}(x, t)$ is meaningful even at the Planck length and set:

$$\Pi_{ij} = -h^{1/2}(K_{ij} - h_{ij}K) = \frac{\delta S}{\delta h^{ij}}, \quad (20)$$

recalling that

$$K_{ij} = -\frac{1}{2N}(\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (21)$$

Hence, as K_{ij} is essentially the time derivative of h_{ij} , equation (20) gives the time evolution of h_{ij} . This time evolution will be different from the classical one due to the presence of the quantum potential in equation (17), which may prevent, among other things, the formation of classical singularities. The notion of spacetime is meaningful in this interpretation, exactly like the notion of trajectory is meaningful in particle quantum mechanics following this interpretation. However, it is not clear if the spacetime geometries constructed from the non-classical solutions $h_{ij}(x, t)$ of equations (17-21) with different choices of $N(x, t)$ and $N_i(x, t)$ will be the same, as in the classical case. This problem will be discussed in more details in the last section.

In the case of homogeneous models, however, the supermomentum constraint \mathcal{H}^i is identically zero, and the shift function N_i can be set to zero in equation (6) without losing any of the Einstein's equations. The hamiltonian (6) is reduced to:

$$H_{GR} = N(t)\mathcal{H}(p^\alpha(t), q_\alpha(t)), \quad (22)$$

where $p^\alpha(t)$ and $q_\alpha(t)$ represent the homogeneous degrees of freedom coming from $\Pi^{ij}(x, t)$ and $h_{ij}(x, t)$. Equations (17-21) become:

$$\frac{1}{2}f_{\alpha\beta}(q_\mu)\frac{\partial S}{\partial q_\alpha}\frac{\partial S}{\partial q_\beta} + U(q_\mu) + Q(q_\mu) = 0, \quad (23)$$

$$Q(q_\mu) = -\frac{1}{R} f_{\alpha\beta} \frac{\partial^2 R}{\partial q_\alpha \partial q_\beta}, \quad (24)$$

$$p^\alpha = \frac{\partial S}{\partial q_\alpha} = f^{\alpha\beta} \frac{1}{N} \frac{\partial q_\beta}{\partial t} = 0, \quad (25)$$

where $f_{\alpha\beta}(q_\mu)$ and $U(q_\mu)$ are the minisuperspace particularizations of G_{ijkl} and $-h^{1/2}R^{(3)}(h_{ij})$, respectively.

Equation (25) is invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same spacetime geometry for a given non-classical solution $q_\alpha(x, t)$.

4 The singularity problem

The question about the persistency of classical cosmological singularities at the quantum level for homogeneous fields has been studied extensively in the literature. In a first approach, the dynamical evolution of the quantum states is obtained by fixing the time gauge before quantization. As we mentioned above, different choices of time gauge imply different quantum theories with different answers to the question we are addressing [2, 3]. In the last section we have shown that this ambiguity in the choice of time does not arise if we apply the causal interpretation to quantum cosmology in the case of minisuperspace models of homogeneous fields. In the present section, we will bring home this fact by making use of a simple minisuperspace example, where the existence of cosmological singularities at the quantum level does not depend on the choice of the time-gauge but only on the choice of the quantum state of the system. This minisuperspace is the Bianchi I model.

The minisuperspace metric is given by:

$$\begin{aligned} ds^2 = & -N^2(t)dt^2 + \exp[2\beta_0(t) + 2\beta_+(t) + 2\sqrt{3}\beta_-(t)] dx^2 + \\ & \exp[2\beta_0(t) + 2\beta_+(t) - 2\sqrt{3}\beta_-(t)] dy^2 + \\ & \exp[2\beta_0(t) - 4\beta_+(t)] dz^2 \end{aligned} \quad (26)$$

The gravitational hamiltonian for this minisuperspace model is:

$$\mathcal{H} = \frac{N}{24 \exp(3\beta_0)}(p_0^2 - p_+^2 - p_-^2). \quad (27)$$

where the p 's are the canonical momenta of the β 's. The classical equations of motion are:

$$p_0^2 - p_+^2 - p_-^2 = 0, \quad (28)$$

$$\dot{\beta}_0 = \frac{\partial \mathcal{H}}{\partial p_0} = \frac{N}{12 \exp(3\beta_0)} p_0, \quad (29)$$

$$\dot{\beta}_+ = \frac{\partial \mathcal{H}}{\partial p_+} = -\frac{N}{12 \exp(3\beta_0)} p_+, \quad (30)$$

$$\dot{\beta}_- = \frac{\partial \mathcal{H}}{\partial p_-} = -\frac{N}{12 \exp(3\beta_0)} p_-, \quad (31)$$

$$\dot{p}_0 = -\frac{\partial \mathcal{H}}{\partial \beta_0} = -\frac{N}{8 \exp(3\beta_0)}(p_0^2 - p_+^2 - p_-^2) = 0, \quad (32)$$

$$\dot{p}_+ = -\frac{\partial \mathcal{H}}{\partial \beta_+} = 0, \quad (33)$$

$$\dot{p}_- = -\frac{\partial \mathcal{H}}{\partial \beta_-} = 0. \quad (34)$$

To discuss the appearance of singularities, we need the Weyl square tensor $W^2 \equiv W^{\alpha\beta\mu\nu}W_{\alpha\beta\mu\nu}$. It is given by:

$$W^2 = \frac{1}{432}e^{-12\beta_0}(-2p_0p_+^3 + 6p_0p_-^2p_+ + p_+^4 + 2p_+^2p_-^2 + p_+^4 + p_0^2p_+^2 + p_0^2p_-^2). \quad (35)$$

Hence, the Weyl square tensor is proportional to $\exp(-12\beta_0)$ because the p 's are constants (see Eqs (32-34)). Solving equation (29) in the gauge $N = 12 \exp(3\beta_0)$, we can see that $\beta_0 = p_0 t$, and the singularity is at $t = -\infty$. It is a fast-time gauge in the terminology of reference [3]. If we choose $N = 1$,

then $\beta_0 = \frac{1}{3} \ln(\frac{p_0}{4}t)$ and the singularity appears at $t = 0$. It is a slow-time gauge. The classical singularity can be avoided only if we set $p_0 = 0$. But then, due to equation (28), we would also have $p_{\pm} = 0$, implying that the Weyl square tensor be identically zero, corresponding to the trivial case of Minkowski spacetime. The conjecture stated in reference [3] says that the singularity persists at the quantum level in the fast-time gauge but disappears in the slow-time gauge.

The Dirac quantization scheme yields the following Wheeler-DeWitt equation:

$$\left(\frac{\partial^2}{\partial \beta_0^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} \right) \Psi = 0. \quad (36)$$

In reference [11], a consistent inner product is constructed, and gauge invariant (Dirac) observables which depends on a parameter, which is nothing but β_0 , are constructed. In this way, the Weyl square observable is built, exhibiting a singularity at $\beta_0 = -\infty$, as in the classical case. As β_0 plays the role of time, this is equivalent to a quantization in the fast-time gauge.

Let us now make use of the causal interpretation. Take the following solution to the Wheeler-DeWitt equation (36):

$$\begin{aligned} \Psi = & \exp[i(\sqrt{k_+^2 + k_-^2} \beta_0 + k_+ \beta_+ + k_- \beta_-)] + \\ & \exp[-i(\sqrt{l_+^2 + l_-^2} \beta_0 + l_+ \beta_+ + l_- \beta_-)] \end{aligned} \quad (37)$$

where the k 's and l 's are real constants. Note that this function is not normalizable, but this is not important for the ontological interpretation. Calculating $\frac{\partial S}{\partial \beta_0}$, $\frac{\partial S}{\partial \beta_+}$, and $\frac{\partial S}{\partial \beta_-}$, where S is the phase of the wave function (37), we obtain:

$$p_0 \equiv \frac{\partial S}{\partial \beta_0} = \frac{1}{2} \sqrt{k_+^2 + k_-^2} - \frac{1}{2} \sqrt{l_+^2 + l_-^2}, \quad (38)$$

$$p_+ \equiv \frac{\partial S}{\partial \beta_+} = \frac{1}{2}k_+ - \frac{1}{2}l_+, \quad (39)$$

$$p_- \equiv \frac{\partial S}{\partial \beta_-} = \frac{1}{2}k_- - \frac{1}{2}l_-. \quad (40)$$

It is easy to see in the above equations that it is possible to have $p_0 = 0$ and $p_{\pm} \neq 0$. We can also understand it by the fact that equation (28) is no longer valid at the quantum level; the quantum potential must be added to it and thus $p_0 = 0$ does not imply $p_{\pm} = 0$. Hence, it is possible to find a curved spacetime without singularities, i.e., a spacetime with a Weyl square tensor which is neither null nor infinite, for the quantized Bianchi I model. Note that this result is independent on the value chosen for N . In particular, we could have chosen the fast-time gauge mentioned previously, and still have a non-singular quantum spacetime. Hence, using the ontological interpretation, we have presented a simple example where the appearance of singularities in the quantum regime depends only on the state of the system, and not on the time-gauge choice we make.

5 Conclusion

In this paper we have shown that the application of the causal interpretation of quantum mechanics to the quantum cosmology of homogeneous fields yields definite predictions without any ambiguity due to the arbitrariness in the time-gauge choice. As a consequence, the slow-fast-time gauge conjecture about the persistency of cosmological singularities at the quantum level is irrelevant within the causal interpretation. Taking the minisuperspace of Bianchi I model, we have shown that the quantum potential of given quantum states can prevent the formation of the classical singularity, yielding a non-trivial regular four-geometry, independently on the choice of the lapse function.

One can argue on why we have obtained non-singular quantum solutions in the quantization of the Bianchi I model while in reference [11] it is shown that all quantum states of this model are singular. The answer is that in order to have a Dirac observable which can play the role of time, Ashtekar *et al.* [11] had to take only positive frequency solutions of equation (36), i.e., states with positive p_0 . In this way, the Dirac observable $\hat{\beta}_0$ becomes proportional to the identity operator, the multiplying constant being time. In the causal interpretation, however, the restriction to positive p_0 is not necessary for having a notion of time: it appears, as in the classical case, via equation (25). Hence, we can construct wave functions which are superpositions of positive and negative frequency solutions as in equation (37), and which does not present any singularity, as was demonstrated in the last section. Note that any superposition of positive and negative solutions is not an eigenstate of the operator $\hat{\beta}_0$ of reference [11]. Indeed, if the Hilbert space is enlarged with the inclusion of negative frequency solutions, we cannot use $\hat{\beta}_0$ as a time operator because it is no longer a multiple of the identity.

A very interesting and fundamental question is about the generalization of this result to the general case of inhomogeneous fields. In this case, the supermomentum constraint \mathcal{H}^i is not identically zero, and the shift function N_i must be present in the hamiltonian (6). The simple time-reparametrization invariant equation (25) will no longer be valid. We have to use the general equations (20) and (21), where S is a solution of the modified Hamilton-Jacobi equation (17).

One can see the problem more clearly by trying to construct a hamiltonian which generates the non-classical evolution of h_{ij} . It would be given by the hamiltonian (6), with \mathcal{H} supplemented by the quantum potential (19):

$$\tilde{H}_{GR} = \int d^3x (N\tilde{\mathcal{H}} + N_j\mathcal{H}^j) \quad (41)$$

where

$$\tilde{\mathcal{H}} = \mathcal{H} - \frac{1}{R} G_{ijkl} \frac{\delta^2 R}{\delta h_{ij} \delta h_{kl}} \quad (42)$$

However, it is not clear if the Poisson brackets of the constraints $\tilde{\mathcal{H}}$ and \mathcal{H}^j close, and it is sure that they do not close like the commutators of the generators of the deformations of three-dimensional spacelike slices cut through a Riemannian spacetime. Indeed, in reference [12] it is shown that the potential term in the super-hamiltonian must be proportional to the scalar curvature of the spacelike slices plus a cosmological term, exactly like in General Relativity, in order that the dynamics of the fields be consistent with the kinematic of deformations. Hence, the dynamics of h_{ij} in the presence of the quantum potential does not satisfy this requirement. This is a very important point which should be investigated further.

Note that, for homogeneous quantum cosmology, the non classical evolution of the homogeneous degrees of freedom can be generated by a hamiltonian with a single constraint,

$$\tilde{H}_{GR} = N(t)\tilde{\mathcal{H}}(p^\alpha(t), q_\alpha(t)), \quad (43)$$

where $\tilde{\mathcal{H}}(p^\alpha(t), q_\alpha(t))$ is the classical constraint supplemented by the quantum potential

$$Q(q_\mu) = -\frac{1}{R}f_{\alpha\beta}\frac{\partial^2 R}{\partial q_\alpha\partial q_\beta}. \quad (44)$$

As a single constraint commutes with itself, the theory is invariant under time reparametrizations and the problems mentioned above do not arise in this restricted case.

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