A Geometric Introduction to Determinants

Consider two vectors in the xy-plane $\mathbf{u} = < u_1, u_2 >$ and $\mathbf{v} = < v_1, v_2 >$ as illustrated in Figure 1 below.

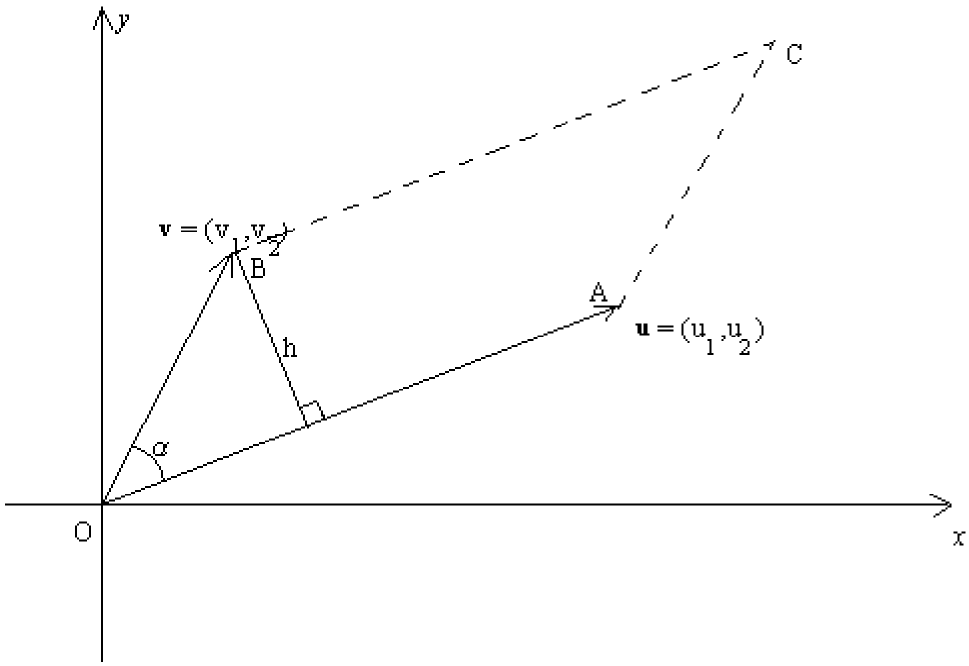

Figure 1

Clearly, the area $\mathcal{A}$ of the parallelogram OACB is the product OA $\times$ h with h $=$ OBsinα. In other words, $\mathcal{A} = \|\mathbf{u}\|\|\mathbf{v}\|\text{sin}\alpha$. In fact, there is an even easier formula for $\mathcal{A}$ using only the components of **u** and **v** as follows:

Proposition 1: $\mathcal{A} = |u_1 v_2 - v_1 u_2|$.

Proof: $\mathcal{A}^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2\text{sin}^2\alpha = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1 - \cos^2\alpha) = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}.\mathbf{v})^2$

$= (u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1 v_1 + u_2 v_2)^2$

$= u_1^2 v_1^2 + u_1^2 v_2^2 + u_2^2 v_1^2 + u_2^2 v_2^2 - u_1^2 v_1^2 - u_2^2 v_2^2 - 2u_1 v_1 u_2 v_2$

$= u_1^2 v_2^2 + u_2^2 v_1^2 - 2u_1 v_1 u_2 v_2 = (u_1 v_2 - v_1 u_2)^2$. QED

Interestingly, the area $\mathcal{A}$ is given by the quantity $(u_1 v_2 - v_1 u_2)$ *up to a sign*. This quantity which we will later call a (2×2) "determinant" indeed changes sign when the order of **u** and **v** is reversed. We will symbolically represent this determinant (involving two vectors of two components, or a 2×2 matrix) by

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - v_1 u_2.$$

Not surprisingly given its geometric interpretation, if one of the two vectors is a scalar multiple of the other, this determinant is nil. Moreover, if one multiplies *one* of the two vectors by a scalar c, the whole determinant is multiplied by that same scalar (since the corresponding area $\mathcal{A}$ is multiplied by the scalar (in absolute value). Somewhat less obvious, but extremely important, is the fact that if one adds a vector **z** $= \; < z_1, z_2 >$ to **u** (for instance), one simply adds the determinant $\begin{vmatrix} z_1 & z_2 \\ v_1 & v_2 \end{vmatrix}$ to $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$. Geometrically, one adds (or subtracts) the area $\mathcal{B}$ of the parallelogram build on **z** and **v** to $\mathcal{A}$.

This approach to the area of a parallelogram can be generalized to the three dimensional case. First, we look at two vectors in $\Re^3$ and extend the above formula for the area $\mathcal{A}$ of the parallelogram they determine. Then we consider the parallelepiped defined by three vectors in space and calculate its volume in similar ways.

In $\Re^3$ we have **u** $= \; < u_1, u_2, u_3 >$ and **v** $= \; < v_1, v_2, v_3 >$. Figure 1 should then involve a third axis y and the picture would be seen in perspective. But the same basic formula $\mathcal{A} =$ OA $\times$ h, with OA $= \|\mathbf{u}\|$, h $=$ OBsinα, and OB $= \|\mathbf{v}\|$, holds. We then have

Proposition 2: $\mathcal{A} = \sqrt{(u_2v_3 - v_2u_3)^2 + (v_1u_3 - u_1v_3)^2 + (u_1v_2 - v_1u_2)^2}$

Proof: Proceed exactly as in Proposition 1 (the algebra is slightly longer and involves the third components).

Note that when all third components are zero, the above formula reduces to that of Proposition 1. However, this new formula looks very much like the norm (length) of a vector, namely:

$< (u_2v_3 - v_2u_3), (v_1u_3 - u_1v_3), (u_1v_2 - v_1u_2) >$

This vector will be called the *cross product* of **u** and **v** and will be denoted **u** $\times$ **v**. It has interesting properties:

1) Recall that the dot product of two perpendicular vectors is nil. One easily finds that $\mathbf{u}.(\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v}.(\mathbf{u} \times \mathbf{v}) = 0$. So, unless any of these vectors is nil, the cross product **u** $\times$ **v** is perpendicular to both **u** and **v**. Therefore, it is perpendicular to the plane spanned by **u** and **v** (provided that plane is well defined, see 2).

2) There are three possible ways in which the cross product **u** $\times$ **v** is the vector **O**. Either $\mathbf{u} = \mathbf{O}$, or $\mathbf{v} = \mathbf{O}$, or **u** and **v** are collinear. Indeed, since the length of **u** $\times$ **v** is the area $\mathcal{A}$, the only way that area is nil is when either side OA or OB is nil or when the angle α is nil or flat (is equal to 0 or π). These three cases are those where there is no plane spanned by **u** and **v**.

3) If one exchanges the order of **u** and **v**, the cross product components all change sign. Therefore $\mathbf{v} \times \mathbf{u} = -\,\mathbf{u} \times \mathbf{v}$.

This brings us to the case of three vectors and the issue of calculating the volume of the parallelepiped they determine. Consider Figure 2 below, which is just Figure 1 with an added vector **w** represented by the segment OD and the parallelepiped with vertices O,A,B,C,D,F,G, and H (not visible).

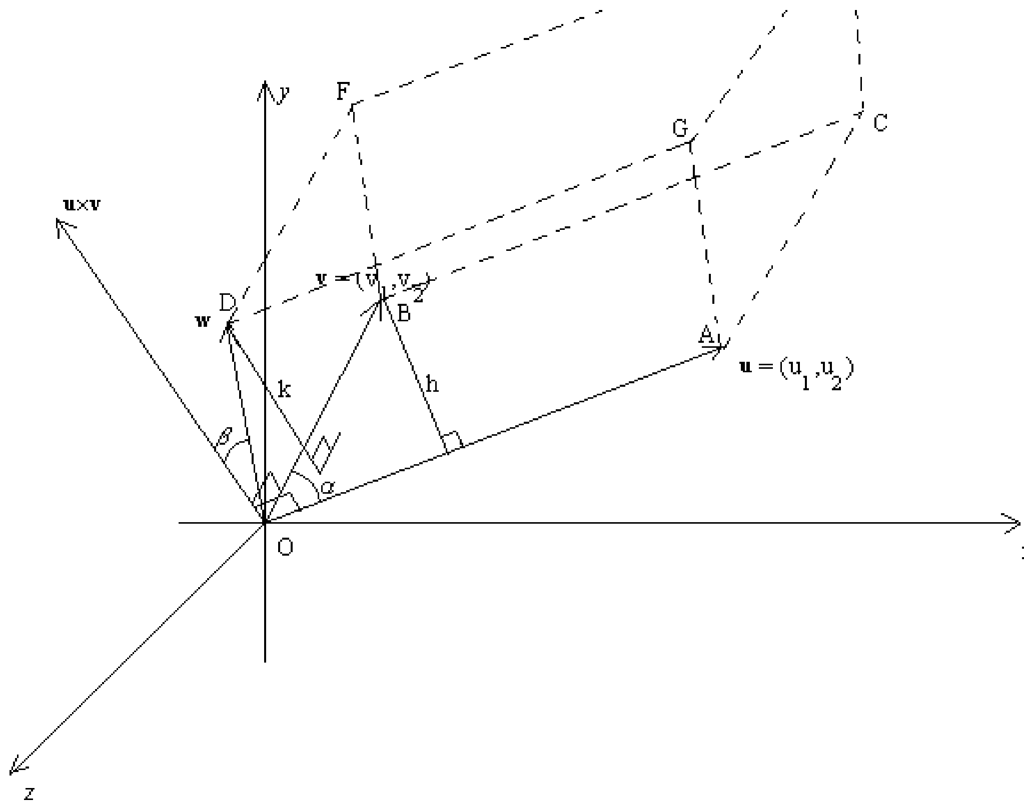

Figure 2

The height k is parallel to the vector $\mathbf{u} \times \mathbf{v}$ and therefore perpendicular to the plane of the parallelogram OACB. However, the new vector $\mathbf{w}$ need not be such. So it makes an angle β with the perpendicular $\mathbf{u} \times \mathbf{v}$. It is therefore clear that $\mathrm{k} = \mathrm{OD}|\cos\beta|$ (the absolute value accounts for the case where β is greater than $\frac{\pi}{2}$). But then, by elementary geometry, the volume $\mathcal{V}$ of the parallelepiped is the area $\mathcal{A}$ multiplied by k. Put another way, it is $\|\mathbf{u} \times \mathbf{v}\|$ multiplied by $\|\mathbf{w}\||\cos\beta|$.

Thus $\mathcal{V} = \|\mathbf{u} \times \mathbf{v}\|\|\mathbf{w}\||\cos\beta|$. But this is simply the absolute value of the dot product $(\mathbf{u} \times \mathbf{v}).\mathbf{w}$ which is called the "triple product" of the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. This triple product has interesting properties:

By direct calculation:

$(\mathbf{u} \times \mathbf{v}).\mathbf{w} = (u_2 v_3 - v_2 u_3) w_1 + (v_1 u_3 - u_1 v_3) w_2 + (u_1 v_2 - v_1 u_2) w_3.$

One also verifies (by exchanging the vectors in the geometric argument) that:

$(\mathbf{u} \times \mathbf{v}).\mathbf{w} = \mathbf{u}.(\mathbf{v} \times \mathbf{w}) =$

$= u_1(v_2 w_3 - w_2 v_3) + u_2(w_1 v_3 - v_1 w_3) + u_3(v_1 w_2 - w_1 v_2).$

Any exchange of any two of the vectors results in a change of sign of the triple product. Evidently, it is nil whenever either $\mathbf{u}$, $\mathbf{v}$, or $\mathbf{w}$, is nil, or when the cross product $\mathbf{u} \times \mathbf{v}$ is nil (meaning $\mathbf{u}$ and $\mathbf{v}$ are collinear), or when $\mathbf{w}$ makes an angle $\beta = \frac{\pi}{2}$ with $\mathbf{u} \times \mathbf{v}$ (meaning it is in the plane determined by $\mathbf{u}$ and $\mathbf{v}$). This triple product is also called a (3×3) "determinant" in what follows.

We will symbolically represent this determinant (involving three vectors of three components, or a 3×3 matrix) as follows:

$$(\mathbf{u} \times \mathbf{v}).\mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Just like in the previous (2 $\times$ 2) determinant, if one multiplies one of the vectors by a scalar c, the determinant is correspondingly multiplied by c. And if one adds a vector $\mathbf{z} = \ < z_1, z_2, z_3 >$ to (say) vector **u**, the determinant $(\mathbf{z} \times \mathbf{v}).\mathbf{w} = \begin{vmatrix} z_1 & z_2 & z_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ is added to $(\mathbf{u} \times \mathbf{v}).\mathbf{w}$. Geometrically this means that the volume $\mathcal{W}$ of the parallelepiped built on **z**, **v**, and **w**, is added to (or subtracted from) $\mathcal{V}$.

We will say that determinants are *linear* in their rows.

There is an interesting property relating the (2×2) and the (3×3) determinants. One can see that:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

In other words, a (3×3) determinant can be "reduced to" three (2×2) determinants by working through the first row and alternating the signs.

More generally, we will define determinants for any $(n \times n)$ matrix as satisfying the following properties (axioms):

I) The determinant is linear in each of the rows;

II) If two rows are identical the determinant is nil (the corresponding surface or volume is nil). Equivalently, one may say that the determinant changes sign when two rows are exchanged;

III) The determinant of the identity matrix is equal to 1.

One can prove that there is a unique determinant thus associated to any $(n \times n)$ matrix. One can also generalize the rules of operation for the calculation of determinants seen above. An $(n \times n)$ determinant can be evaluated using n smaller $(n-1) \times (n-1)$ determinants as above with a careful alternation in sign. We will accept those two facts without proof.

At that point, all properties of determinants can be deduced from the above three axioms. Let us list a few:

If E_{ij} is an elementary matrix (representing a row addition in Gaussian elimination), then its determinant is 1. If a matrix is diagonal, then its determinant is the product of diagonal elements. More generally, if a matrix is triangular, its determinant is the product of the diagonal elements.

A permutation matrix has a determinant of ± 1.

The determinant of a matrix product AB is the product of the determinants of A and B.

A matrix is nonsingular (thus invertible) if and only if its determinant is non-zero.

The determinant of A^{T} is the same as the determinant of A.