GameTheory

Game theory is usually considered the brainchild of John von Neumann (1928) but it finds its roots in the work of Augustin Cournot (1838), and the major figure is probably John Nash (1950). Game theory is concerned with the mathematical problems of rationality when two or more decision makers can influence an outcome but have different and possibly conflicting priorities. The analysis begins with the definition of *players*, *strategies*, and *utilities*, and then goes on with that of (Nash) *equilibrium* and the basic existence proof (by Nash). In this first lecture, I will present the basic ideas along with some examples.

Definitions:
1) There is a (finite) set N of n ($n \geq 2$) players;
2) For each player $i \in N$ there is a (finite) set S_i of strategies;
3) For each player $i \in N$ there is a (utility) function $U_i : S = \times_{i \in N} S_i \to \Re$.

Example 1: Consider the well-known two player game of "matching pennies." For each player there are two possible strategies in the "one-shot" game: heads (H) or tails (T). If player 1 is the one who wins when the pennies match, then we have the following (natural) utility function for 1:

$$U_1(H,H) = U_1(T,T) = 1, \qquad U_1(H,T) = U_1(T,H) = -1$$

In fact, this is a so-called "zero sum" game where what one player wins is what the other loses. So $U_2(X_1, X_2) = -U_1(X_1, X_2)$ for any strategy pair $(X_1, X_2) \in S$.

A very convenient representation of the utilities is the table (matrix) form

U_1, U_2	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

where Player 1 (Row) chooses H or T and Player 2 (Column) chooses H or T. The resulting utilities are given in the corresponding cells.

Example 2: Recall the famous Prisoner's Dilemma mentioned in Class #1. Its matrix representation is (with abbreviations Coop for cooperate and Dfct for defect)

U_1, U_2	$Coop$	$Dfct$
$Coop$	$0, 0$	$-2, 1$
$Dfct$	$1, -2$	$-1, -1$

Assumptions:
1) Players can randomize their choices: Player i can choose a *mixed* strategy $X_i \in \mathfrak{X}_i$ where $\mathfrak{X}_i$ is the set of distributions over the set S_i;
2) A player's utility U_i from randomized choices is defined as the mathematical expectation of the utilities of the randomized choices;
3) Each player i seeks, through the choice of $X_i \in \mathfrak{X}_i$, to maximize the expected utility U_i given the choices $X_{-i} \in \underset{j \neq i}{\times} S_j$ by all other players.

Let us return to the two above examples. In the game of matching pennies, if Column is expected to choose H then Row should choose H. But in that expectation, Column should choose T which would incite Row to choose T and would prompt Column to return to H! Players following this thought process appear to cycle in their decision making process.

In the Prisoner's Dilemma instead, Row can immediately see that Dfct is better than Coop no matter what Column does. We say that Dfct *strictly dominates* Coop. In particular, each side is doing its best by choosing Dfct while expecting Dfct to be chosen by the opponent.

In general, what the opponent can be expected to rationally do is critical to what one can rationally do. In the game of matching pennies, if Column is expected to choose H or T each with probability $\frac{1}{2}$, then Row becomes *indifferent* between choosing H or T. That, in particular allows Row to choose H or T each with probability $\frac{1}{2}$ *optimally*. Since the same holds true for Column, the two sides find themselves in an "equilibrium" state where they are maximizing their utility given the other side's similar behavior.

We formalize this situation as follows:

Definition:
A point $X^* \in S$ such that, for each $i \in N$, X_i^* maximizes $U_i(X_i, X_{-i}^*)$ is called a (Nash) equilibrium.

$Theorem$: There always exists a (Nash) equilibrium (in mixed strategies).

The proof that I now outline is the original one due to Nash in 1950. It reduces the existence of equilibrium problem to that of a fixed point of a continuous map from a compact and convex subset of Euclidean space into itself and makes use of the Brouwer Fixed Point Theorem.

First, it is clear that $\mathfrak{X} = \underset{i \in N}{\times} \mathfrak{X}_i$ is a closed, bounded, and convex subset of $\Re^m$ where $m = \underset{i \in N}{\times} m_i$ and m_i is the cardinal of S_i. To define a continuous map f from $\mathfrak{X}$ into itself, we proceed as follows:

For any given $X \in \mathfrak{X}$ and for any $s \in S_i$, we let

$$\sigma_s(X) = \max\{0, U_i(s, X_{-i}) - U_i(X)\} \geq 0$$

Clearly, σ_s is continuous in X by definition of U_i as an expected utility (thereby multilinear in the X_j's). Now, if x_s is the probability of s in the distribution X_i, we let

$$\tau_s(X) = \frac{x_s + \sigma_s(X)}{1 + \sum_{s \in S_i} \sigma_s(X)}$$

By construction $\tau_s(X)$ is continuous in X, $\tau_s(X) \geq 0$, and $\sum_{s \in S_i} \tau_s(X) = 1$. So, the map f that associates the vector of s-component $\tau_s(X)$ to the vector X is continuous and maps $\mathfrak{X}$ into itself. It must therefore (by Brouwer) have a fixed point X^*.

We now verify that X^* is indeed a Nash equilibrium. Let us pick one arbitrary player i. Among the $s \in S_i$ such that $x_s^* > 0$, there must be at least one such that $\sigma_s(X^*) = 0$ since $U_i(X) = \sum_{s \in S_i} x_s U_i(s, X_{-i})$ by definition. For that s we have

$$x_s^* = \frac{x_s^*}{1 + \sum_{s \in S_i} \sigma_s(X^*)}$$

Therefore $\sum_{s \in S_i} \sigma_s(X^*) = 0$ and *all* $\sigma_s(X^*) = 0$ (for all $s \in S_i$ for all i). But this means that no (mixed) strategy Y_i is better than X_i^* against X_{-i}^* for any i. So X^* is a Nash equilibrium. *Q.E.D.*

Calculating Nash equilibria is not a simple algebraic exercise. In the two-person zero-sum case, it is not too difficult to show that the equilibrium is the solution of a linear programming problem for which powerful methods are known (such as the Simplex Algorithm). For two-person non-zero-sum games, a technique known as Complementary pivot programming is known to succeed. For three or more -person game, the problem is essentially open. In practice, one proceeds in two main steps:

1) eliminate all dominated strategies (if any);

2) solve linear equations that make one side indifferent between strategies whose weight is positive in the symmetric equations that make the other side indifferent... Not always a pleasant search. Beyond 3×3 matrices, this can be quite time consuming unless well guided by intuition.

Homework

1) Consider the well-known *Rock-Scissors-Paper* game where *Rock* (Strategy 1) beats *Scissors* (Strategy 2) beats *Paper* (Strategy 3) which beats $Rock$. Assume players receive 1 for win, 0 for tie, and -1 for loss. Write the matrix form of the game and finds its Nash equilibrium.