Markov Chains (continued)

We studied earlier the concept of *hitting time* $H_i^j = \inf\{n \geq 0 : X_0 = i,\ X_n = j\}$ (the minimum time it takes X_n to hit j starting from i). A somewhat similar concept is that of *first return time* $T_i = \inf\{n > 0 : X_0 = i,\ X_n = i\}$ (also a random variable). This is clearly related to the probability $\mathrm{p}_{ii}^{(n)}$ discussed earlier (the ii entry in T^n). The Markov property we discussed earlier extends as follows

Assume $(X_n)_{n\geq 0}$ is Markov (λ,T). Then, conditional on $T_i < \infty$, $(X_{T_i+n})_{n\geq 0}$ is Markov (δ_i,T) (and independent of prior events). This fact extends to all "stoping times" including the m^{th} return time that we will define below.

A probability related to this first return time is the so-called *return probability* $f_i = P(T_i < \infty | X_0 = i)$. To understand the important relation between f_i and the $\mathrm{p}_{ii}^{(n)}$ we need some additional definitions and some lemmata.

We let V_i be the *number of visits* (of the random variable X_n) to i. It can be formally written $V_i = \sum_{n=0}^{\infty} 1_{\{X_n = i\}}$ where $1_{\{X=i\}}$ is the *function* equal to 1 when $X = i$ and to 0 otherwise. *Conditional on* starting at $X_0 = i$, the expected number of visits to i is therefore easily calculated as

$$\begin{aligned} E(V_i) &= E\Big(\sum_{n=0}^{\infty} 1_{\{X_n=i\}}\Big) = \sum_{n=0}^{\infty} E(1_{\{X_n=i\}}) \\ &= \sum_{n=0}^{\infty} P(X_n = i | X_0 = i) = \sum_{n=0}^{\infty} \mathrm{p}_{ii}^{(n)} \end{aligned} \tag{1}$$

However, the expectation of a non-negative integer-valued random variable has another interpretation as follows:

$$\begin{aligned} E(V) &= \sum_{n=1}^{\infty} n P(V = n) = \sum_{n=1}^{\infty} \Big(\sum_{m=0}^{n-1} P(V = n) \Big) \\ &= \sum_{m=0}^{\infty} \Big(\sum_{n=m+1}^{\infty} P(V = n) \Big) = \sum_{m=0}^{\infty} P(V > m) \end{aligned} \tag{2}$$

So, (1) and (2) together show that (with both sides finite or infinite)

$$\sum_{m=0}^{\infty} P(V_i > m) = \sum_{n=0}^{\infty} \mathrm{p}_{ii}^{(n)} \tag{3}$$

But the probabilities $P(V_i > m)$ can be written in terms of f_i using the Markov property and an additional definition: Let $T_i^{(m)}$ be the m^{th} return time to i (starting from $X_0 = i$) defined recursively by $T_i^{(0)} = 0$ (since you are already there) and

$$T_i^{(m+1)} = \inf\{n > T_i^{(m)} : X_n = i\}$$

The interval $S_i^{(m+1)} = T_i^{(m+1)} - T_i^{(m)}$ in fact behaves just like the first return time so we can write $P(S_i^{(m+1)} < \infty | T_i^{(m)} < \infty) = f_i$. Thus, according to the Markov property

$$P(V_i > m+1) = P(T_i^{(m+1)} < \infty) = P(S_i^{(m+1)} < \infty, T_i^{(m)} < \infty)$$

$$= f_i P(T_i^{(m)} < \infty) = f_i P(V_i > m) \qquad (4)$$

Since $P(V_i > 0) = 1$, we must have $P(V_i > m) = f_i^m$.

We can now put things together as follows:

$$\sum_{m=0}^{\infty} P(V_i > m) = \sum_{m=0}^{\infty} f_i^m = \sum_{n=0}^{\infty} \mathrm{p}_{ii}^{(n)} \qquad (5)$$

So, only two things can happen

1) $f_i = 1$: Then $\lim_{m \to \infty} P(V_i > m) = P(V_i = \infty) = 1$ and the probability of infinitely many visits to i is equal to 1. Moreover $\sum_{n=0}^{\infty} \mathrm{p}_{ii}^{(n)} = \infty$. Such a state i is called *recurrent*.

2) $f_i < 1$: Then $\lim_{m \to \infty} P(V_i > m) = P(V_i = \infty) = 0$ and the probability of infinitely many visits to i is nil. Moreover $\sum_{n=0}^{\infty} \mathrm{p}_{ii}^{(n)} < \infty$. Such a state i is called *transient*.

The concepts of recurrence and transience extend to communicating classes that we mentioned earlier. Suppose i and j belong to a same communicating class. We can say something important about $\mathrm{p}_{jj}^{(n)}$: there must exist q and r *fixed* such that $\mathrm{p}_{ij}^{(q)}$ and $\mathrm{p}_{ji}^{(r)}$ are both positive. But then $\mathrm{p}_{jj}^{(n)} = \mathrm{p}_{ji}^{(r)} \mathrm{p}_{ii}^{(m)} \mathrm{p}_{ij}^{(q)}$ for any $n = r + m + q$. It follows that $\sum_{n=0}^{\infty} \mathrm{p}_{jj}^{(n)}$ has the same property of being finite or infinite as $\sum_{n=0}^{\infty} \mathrm{p}_{ii}^{(n)}$. So, a whole class must be either recurrent or transient.

In the finite case, there is a nice characterization. We say that a class is *closed* if there is no escape from it. then a class is recurrent if and only if it is closed (see Theorems 1.5.5 and 1.5.6 p. 27 of Norris).

Homework

1) Problem #1.2.1 p. 11 of Norris.

2) Problem #1.5.1 p. 28 of Norris.