Markov Chains (continued)

A typical application of the theorems of the last class is the phenomenon known as "Gambler's ruin." Suppose that you gamble \$1 at a time with a probability q of losing your bet and p of doubling it (i.e., keeping it and winning \$1). What is the probability that, entering with an initial endowment of $\$i$, you leave broke?

Let h_i be the probability that, starting with $\$i$, you hit \$0. Clearly

$h_0 = 1$, since you are already broke, and
$$h_i = ph_{i+1} + qh_{i-1} \quad \text{for } i = 1., 2, .. \tag{1}$$

The equation (1) is known as a "difference" equation. It is somewhat similar to a differential equation and can be solved with similar methods. Let us try a solution of the form $h_i = \lambda^i$. Replacing in (1) yields

$\lambda^i = p\lambda^{i+1} + q\lambda^{i-1}$ or, for $i \geq 1$, dividing through by λ^{i-1}

$$\lambda = p\lambda^2 + q$$

which is the "characteristic" (quadratic) equation. Solving yields $\lambda = 1$, and $\lambda = \frac{q}{p}$. The "general" solution of (1), if $p \neq q$, is

$$h_i = \text{a} + \text{b}\left(\frac{q}{p}\right)^i \tag{2}$$

If $q > p$, boundedness requires that $\text{b} = 0$ and the initial condition implies $\text{a} = 1$. So, $h_i = 1$ (for all $i \geq 0$) and a gambler is guaranteed to end up broke. If $q = p$ (a very fair game), $\lambda = 1$ is double root and the solution takes the form

$$h_i = \text{a} + \text{b}i$$

Again, boundedness requires $\text{b} = 0$ and the boundary condition implies $\text{a} = 1$. Again, the gambler is sure to end up broke.

If $p > q$ then (2) can be written (with $\text{b} = 1 - \text{a}$)

$$h_i = \left(\frac{q}{p}\right)^i + \text{a}\left(1 - \left(\frac{q}{p}\right)^i\right) \tag{3}$$

This is where the minimality condition of the theorem comes into play and yields a $= 0$. So, we get $h_i = \left(\frac{q}{p}\right)^i$ and the probability of ending broke decreases geometrically with the gamblers initial fortune (but is never zero).

We may also inquire about the mean time to bankrupcy. The equation for mean times takes the form

$k_0 = 0$, since you are already broke, and
$$k_i = 1 + pk_{i+1} + qk_{i-1} \qquad \text{for } i = 1., 2, .. \tag{4}$$

This is now a non-homogeneous difference equation which can be solved by the method of superposition. If we know one particular solution of (4), then the general solution is obtained by adding to it the general solution of the homogeneous case. Let us try the particular solution of (4) $k_i = \alpha i$. We obtain

$$\alpha i = 1 + p\alpha(i+1) + q\alpha(i-1)$$

or

$$0 = 1 + \alpha(p - q)$$

which yields $\alpha = \frac{1}{q-p}$ for $q \neq p$. So, $k_i = \frac{i}{q-p}$ is a particular solution of (4). The general solution of the homogeneous equation is the same as that of (1). For $q \neq p$ we get (with constants a and b to be identified)

$$k_i = \frac{i}{q-p} + \mathrm{a} + \mathrm{b}\left(\frac{q}{p}\right)^i \tag{5}$$

Since $k_0 = 0$ we get $\mathrm{a} + \mathrm{b} = 0$. If $q > p$, we let $k_i = \frac{i}{q-p} + \mathrm{b}\left(\left(\frac{q}{p}\right)^i - 1\right)$. The minimal non-negative solution is therefore $k_i = \frac{i}{q-p}$. If $q < p$, let instead

$$k_i = -\frac{i}{p-q} + \mathrm{a}\left(1 - \left(\frac{q}{p}\right)^i\right)$$

This can never be finite and non-negative for all i. This just means that $k_i = \infty$ for all $i \geq 1$.

Homework

1. In the case $p = q$ above, calculate the mean time to bankrupcy. Hint: try the particular solution $k_i = \alpha i^2$.