Some More Linear Algebra

Until now, we have dealt exclusively with matrices that were diagonalizable. A sufficient condition was that all eigenvalues be distinct (of course, this is not necessary since the identity matrix has the repeated eigenvalue 1 and is already in diagonal form). The chances of hitting repeated eigenvalues when selecting the coefficients of a matrix at random was seen to be negligible (at least for hermitian matrices). But sometimes, natural examples lead to repeating eigenvalues. For instance, the transition matrix

$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

has eigenvalues $\lambda = 1$ (single) and $\lambda = -\frac{1}{2}$ (double). Yet, it appears naturally in a Markov chain problem (Norris, #1.1.4). How does one then deal with the calculation of T^n in that case? The answer is the so-called Jordan form that I will only outline.

We define a Jordan block J_r as a $r \times r$ matrix of the form $J_r = \lambda I_r + N_r$ where λ is some scalar, I_r is the $r \times r$ identity matrix, and N_r is a $r \times r$ made up of 0 everywhere *except* for the "first superdiagonal" that is entirely made up of 1's. For instance, for $r = 3$

$$N_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

A matrix such as N_r has the following interesting property: N_r^k (for $k < r$) has zeros everywhere except for the k^{th} superdiagonal that is entirely made up of 1's. For instance

$$N_3^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In fact, one finds that for $k \geq r$, $N_r^k = 0$. Such a matrix is called "nilpotent." Furthermore, it is obvious that $I_r N_r = N_r I_r$. But this has an interesting consequence: Suppose we want to calculate $(\lambda I_r + N_r)^n$. In general, the product of matrices does not commute, but here the do. So, we can directly apply the binomial theorem and get

$$(\lambda I_r + N_r)^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} N_r^k$$

Now, suppose that a matrix A is made up of Jordan blocks along its diagonal, and zeros elsewhere. For instance

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

has a (trivial) Jordan block $\begin{pmatrix} 1 \end{pmatrix}$ and a (less trivial) Jordan block $\begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}$. A matrix such as A is said to be "in Jordan form." The interesting fact is that what we said of Jordan blocks extends to such a Jordan form matrix. The reason is that whatever multiplication occurs on one block has no effect on any other block. So, the binomial theorem extends and reads

$$(\Lambda + N)^n = \sum_{k=0}^{n} \binom{n}{k} \Lambda^{n-k} N^k \tag{1}$$

Moreover, since any block-component of N is nilpotent for $k = r$, where r is the multiplicity of the corresponding eigenvalue, N is also nilpotent with k being the maximum multiplicity of any of its eigenvalues. The matrix A above is clearly such that $N^2 = 0$ and the binomial formula (1) reduces to (at most) two terms

$$(\Lambda + N)^n = \Lambda^n + n\Lambda^{n-1} N$$

In practical applications, this means that an eigenvalue λ of multiplicity r will be involved with a polynomial of degree up to $r - 1$ in the expansion of $(\Lambda + N)^n$.