Markov Chains (continued)

One important issue arises when in the method discussed in the last class when the eigenvalues of $T$ include complex numbers. How do we interpret the effect of such eigenvalues on probabilities that are real by definition? Let us discuss the following example

\[ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \]

The eigenvalues are the solutions of $p(\mu) = -\mu(\mu - \frac{1}{3}) + \frac{1}{4} = 0$, or $\mu = 1, \pm \frac{i}{2}$. Suppose we are interested in the probability of being at state 1, starting from state 1, after $n$ turns. In the notations of Norris, this reads $p_{11}^{(n)}$. Using the same argument as in the last class, this must read

\[ p_{11}^{(n)} = a + b\left(\frac{i}{2}\right)^n + c\left(-\frac{i}{2}\right)^n \tag{1} \]

However, by Euler's formula, $\left(\pm\frac{i}{2}\right)^n = \left(\frac{1}{2}\right)^n \left(\cos \frac{n\pi}{2} \pm i\sin \frac{n\pi}{2}\right)$. Combining with (1) yields

\[ p_{11}^{(n)} = a + \left(\frac{1}{2}\right)^n \left\{ (b + c)\cos \frac{n\pi}{2} + i(b - c)\sin \frac{n\pi}{2} \right\} \tag{2} \]

So, if $b$ and $c$ are complex conjugates, we can write $\beta = (b + c)$ and $\gamma = i(b - c)$, both real, and the expression in (2) is real. The same technique as before applies. We write

\[ p_{11}^{(0)} = a + \beta = 1 \quad \text{(since we start at state 1)} \]
\[ p_{11}^{(1)} = a + \frac{1}{2}\gamma = 0 \quad \text{(since the } p_{11} = 0) \]
\[ p_{11}^{(2)} = a - \frac{1}{4}\beta = 0 \quad \text{(by calculating } \lambda T^2) \]

It follows that $a = \frac{1}{5}$, $\beta = \frac{4}{5}$, and $\gamma = -\frac{2}{5}$. In particular, the long run probability of being at state 1 is $\frac{1}{5}$.

In general, we can calculate the probability $p_{ij}^{(n)}$ of reaching state $j$ from state $i$ after $n$ turns in the very same way. We can then ask whether it is possible to reach state $j$ from state $i$ which, of course, requires that $p_{ij}^{(n)} > 0$ for some $n$. We will say that $i$ leads to $j$ (denoted $i \rightarrow j$) whenever that probability is positive for some $n$. We similarly say that $i$ communicates with $j$ (denoted $i \leftrightarrow j$) when both $i \rightarrow j$ and $j \rightarrow i$. This is clearly an
equivalence relation that defines so-called communicating classes. Such a class is closed if whenever \( i \to j \) and \( i \) is in the class then \( j \) is in the class (in other words, you cannot leave a closed class). A state \( i \) that forms a closed class is called absorbing. If a Markov chain is such that \( I \) is a single class, then it is called irreducible.

We can also define the hitting time \( H_i^j \) as the minimum time \( n \) it takes for \( X_n \) to be in state \( j \), starting from \( X_0 = i \) (with the convention that this is \( \infty \) if \( j \) can never be reached from \( i \)). This is clearly a random variable. We can similarly define \( H_i^A \) for a subset \( A \subseteq I \). We may now define the probability \( h_i^A = P(H_i^A < \infty) \) that, starting from state \( i \), \( (X_n)_{n \geq 0} \) ever hits \( A \). When \( A \) is a closed class, this will be called its absorption probability.

With the example of last time, it is clear that the whole set \( I = \{1,2,3\} \) forms an irreducible chain since all states communicate with each other. Therefore, the hitting probabilities \( h_i^A \) are all trivially 1. A less trivial example is given by the following diagram (from Norris p. 12)

\[
\begin{align*}
\text{Clearly, the probability, starting from state 1, that the chain ever hits anything but state 1 is nil. But what is, for instance, the probability that starting from 2, the chain hits (and is absorbed into) 4?} \ h_2^4 \text{ denotes the probability that, starting from } i, \text{ the chain hits 4. Clearly, } h_1^4 = 0 \text{ and } h_3^4 = 1. \text{ Moreover:} \\
&h_2^4 = \frac{1}{2} h_1^4 + \frac{1}{2} h_3^4 \\
&h_3^4 = \frac{1}{2} h_2^4 + \frac{1}{2} h_3^4
\end{align*}
\]

Solving (3) yields \( h_2^4 = \frac{1}{3} \).

A related question is how long it takes to reach 4 starting from 2. In other words, what is the mean time in which one reaches \( j \) from \( i \)? This is simply the expectation of the random variable \( H_i^j \) that reads (and may be infinite)

\[
k_i^j = E(H_i^j) = \sum_{n < \infty} n P(H_i^j = n) + \infty P(H_i^j = \infty)
\]

With the same example, we find \( k_1^4 = \infty \). Moreover \( k_2^4 = 1 + \frac{1}{2} k_1^4 + \frac{1}{2} k_3^4 = \infty \). So, the expected time it takes to reach 4 from 1 is infinite because of the possibility to be
absorbed in 1 from which it become impossible to reach 4. If, however, we ask the mean
time of reaching \(A = \{1, 4\}\) from 2 we have \(k_1^A = k_4^A = 1\) and

\[
\begin{align*}
k_2^A &= 1 + \frac{1}{2}k_1^A + \frac{1}{2}k_3^A \\
k_3^A &= 1 + \frac{1}{2}k_2^A + \frac{1}{2}k_4^A
\end{align*}
\]

which yields \(k_2^A = 2\).

More generally, we can obtain the following

**Theorem 1:** The vector of hitting probabilities \(h^A = (h_i^A : i \in A)\) is the minimal
non-negative solution of the system of linear equations

\[
\begin{align*}
h_i^A &= 1 & \text{for } i \in A \\
h_i^A &= \sum_{j \in I} p_{ij}h_j^A & \text{for } i \notin A
\end{align*}
\]

**Proof:** Norris, pp. 13-14.

**Theorem 2:** The vector of mean hitting times \(k^A = (k_i^A : i \in I)\) is the minimal
non-negative solution of the system of linear equations

\[
\begin{align*}
k_i^A &= 0 & \text{for } i \in A \\
k_i^A &= 1 + \sum_{j \notin A} p_{ij}k_j^A & \text{for } i \notin A
\end{align*}
\]

**Proof:** Norris, pp. 17-18.

**Homework**

1. Problem 1.3.2 p. 18 of Norris.