Random Matrices (continued)

We now investigate the consequences of our findings on the probability distribution of eigenvalues of an arbitrary 2×2 hermitian matrix. Here again, the results would generalize to the $n \times n$ case, but we will get a sense of the issues with the 2×2 case. The basis for our approache is the fact that the trace of any matrix is simply the sum of its eigenvalues (counting multiplicity if they are repeated). So, if λ and μ are the eigenvalues of H, the eigenvalues of H^2 are clearly λ^2 and μ^2 and $\mathrm{trace}(H^2) = \lambda^2 + \mu^2$. So, we can consider a change of variables from x_{11}, x_{12}, y_{12}, and x_{22} to λ, μ and other variables (α and β below) involved in the diagonalization of H.

From Homework Problem #2 in Class #7, we know that the most general 2×2 unitary matrix U can be written (up to a factor $e^{i\theta}$):

$$U = \begin{pmatrix} \cos\alpha & -(\sin\alpha)e^{i\beta} \\ (\sin\alpha)e^{-i\beta} & \cos\alpha \end{pmatrix} \qquad (1)$$

with $\alpha \in [0, \frac{\pi}{2})$ and $\beta \in [0, 2\pi)$. However, these will depend on the order in which λ and μ are placed in the diagonal of Λ. So, we may make the convention that $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda \geq \mu$. In that case, α and β in (1) are uniquely determined and we may proceed confidently with a change of variable. We may write (with $dH = dx_{11}dx_{22}dx_{12}dy_{12}$):

$$\int\int\int\int_{\mathcal{A}} e^{-\mathrm{trace}(H^2)} dH = \int\int\int\int_{\mathcal{B}} e^{-(\lambda^2+\mu^2)} |J(\lambda,\mu,\alpha,\beta)| d\lambda d\mu d\alpha d\beta \qquad (2)$$

where $J(\lambda,\mu,\alpha,\beta) = \frac{\partial(x_{11},x_{22},x_{12},y_{12})}{\partial(\lambda,\mu,\alpha,\beta)}$ is the usual Jacobian determinant. Calculating this determinant is what will occupy us now. Evidently, the entries of J can be found in those of $\frac{\partial H}{\partial \lambda}, \ldots, \frac{\partial H}{\partial \beta}$. The only question is how to extract them in usable form. To do so, let us investigate some consequences of

$$U^* H U = \Lambda \qquad (3a)$$

with

$$U^* U = I \qquad (3b)$$

Differentiating (3a) with respect to λ and μ yields $U^* \frac{\partial H}{\partial \lambda} U$ and $U^* \frac{\partial H}{\partial \mu} U$ on the left and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ on the right of (3a) respectively (since U is independent of $\lambda$and μ). Differentiating with respect to α and β requires a little more caution. Let us write (3a) as $H = U\Lambda U^*$ and let us differentiate (say with respect to α) to reach

$$\frac{\partial \mathrm{H}}{\partial \alpha} = \frac{\partial \mathrm{U}}{\partial \alpha}\Lambda \mathrm{U}^* + \mathrm{U}\Lambda\frac{\partial \mathrm{U}^*}{\partial \alpha} \tag{4a}$$

or

$$\mathrm{U}^*\frac{\partial \mathrm{H}}{\partial \alpha}\mathrm{U} = \left(\mathrm{U}^*\frac{\partial \mathrm{U}}{\partial \alpha}\right)\Lambda + \Lambda\left(\frac{\partial \mathrm{U}^*}{\partial \alpha}\mathrm{U}\right) = \mathrm{A}\Lambda + \Lambda\mathrm{A}^* \tag{4b}$$

Now, because of (3b) we have $\mathrm{U}^*\frac{\partial \mathrm{U}}{\partial \alpha} + \frac{\partial \mathrm{U}^*}{\partial \alpha}\mathrm{U} = 0 = \mathrm{A} + \mathrm{A}^*$. So:

$$\mathrm{U}^*\frac{\partial \mathrm{H}}{\partial \alpha}\mathrm{U} = \mathrm{A}\Lambda - \Lambda\mathrm{A} = (\lambda-\mu)\begin{pmatrix} 0 & \mathrm{s}_{12}^{\alpha} \\ \mathrm{s}_{21}^{\alpha} & 0 \end{pmatrix} \tag{4c}$$

Differentiating with β is similar to α. So we have all four matrices for $\mathrm{U}^*\frac{\partial \mathrm{H}}{\partial \xi}\mathrm{U}$ with $\xi = \lambda$, μ, α, and β. We will now extract from those matrices the jacobian we want. We first remark that for any of our four cases of variable ξ, the entry (i,j) of $\mathrm{U}^*\frac{\partial \mathrm{H}}{\partial \xi}\mathrm{U}$ (let us denote it $(\mathrm{U}^*\frac{\partial \mathrm{H}}{\partial \xi}\mathrm{U})_{ij}$) is obtained from the same linear combination of the entries of $\frac{\partial \mathrm{H}}{\partial \xi}$. For instance:

$$\begin{aligned}(\mathrm{U}^*\frac{\partial \mathrm{H}}{\partial \xi}\mathrm{U})_{ij} &= l_{11}^{ij}\frac{\partial \mathrm{x}_{11}}{\partial \xi} + l_{22}^{ij}\frac{\partial \mathrm{x}_{22}}{\partial \xi} + l_{12}^{ij}\frac{\partial \mathrm{x}_{12}}{\partial \xi} + l_{21}^{ij}\frac{\partial \mathrm{y}_{12}}{\partial \xi} \\ &= (\tfrac{\partial \mathrm{x}_{11}}{\partial \xi},\tfrac{\partial \mathrm{x}_{22}}{\partial \xi},\tfrac{\partial \mathrm{x}_{12}}{\partial \xi},\tfrac{\partial \mathrm{y}_{12}}{\partial \xi})(l_{11}^{ij},l_{22}^{ij},l_{12}^{ij},l_{21}^{ij})^T \end{aligned} \tag{5}$$

So, if we let $\mathrm{L}_{ij} = (l_{11}^{ij},l_{22}^{ij},l_{12}^{ij},l_{21}^{ij})^T$ and let L be the detrerminant with columns L_{ij} as follows: $\mathrm{L} = |\mathrm{L}_{11},\mathrm{L}_{22},\mathrm{L}_{12},\mathrm{L}_{21}|$, we can easily write the *product* determinant

$$\mathrm{JL} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{s}_{12}^{\alpha}(\lambda-\mu) & \mathrm{s}_{21}^{\alpha}(\lambda-\mu) \\ 0 & 0 & \mathrm{s}_{12}^{\beta}(\lambda-\mu) & \mathrm{s}_{21}^{\beta}(\lambda-\mu) \end{vmatrix} = (\lambda-\mu)^2\begin{vmatrix} \mathrm{s}_{12}^{\alpha} & \mathrm{s}_{21}^{\alpha} \\ \mathrm{s}_{12}^{\beta} & \mathrm{s}_{21}^{\beta} \end{vmatrix} = \mathrm{S}(\lambda-\mu)^2$$

Now, clearly, all the elements of L and S depend solely on α and β (not on λ and μ). So, we can write $|\mathrm{J}| = \frac{|\mathrm{S}|}{|\mathrm{L}|}(\lambda-\mu)^2 = g(\alpha,\beta)(\lambda-\mu)^2$. Replacing in (2) yields

$$\begin{aligned}&\int\int\int\int_{\mathcal{B}} e^{-(\lambda^2+\mu^2)}|\mathrm{J}(\lambda,\mu,\alpha,\beta)|\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\alpha\mathrm{d}\beta \\ &= \mathrm{c}\int\int_{\mathcal{B}} e^{-(\lambda^2+\mu^2)}(\lambda-\mu)^2\mathrm{d}\lambda\mathrm{d}\mu\end{aligned} \tag{6}$$

where c is a normalizing constant that includes the integration of $g(\alpha,\beta)$.

Formula (6) shows the form of the probability density function of the eigenvalues λ and μ for a 2×2 hermitian matrix. The entire above construction can be generalized to the $n\times n$ case and reads:

$$c_n \int\int_{\mathcal{B}} e^{-\sum_{i=1}^{n} \lambda_i^2} \prod_{j<i} (\lambda_i - \lambda_j)^2 d\lambda_1 ... d\lambda_n \qquad (7)$$

The square factor characterizes the hermitian case. Other cases involve other powers of the difference of eigenvalues.

Homework

1. In (6), given λ, find μ where the probability density is maximum. Write an expression for the mean value of μ (do not estimate it).