Linear Algebra (continued)

We now investigate the eigenvectors associated to the eigenvalues found last time. First, consider $\lambda_1 = 2$. Let $v_1^T = (v_{11}, v_{21})$ be a normal eigenvector (any multiple of an eigenvector is still an eigenvector). It must satisfy $Av_1 = 2v_1$ or, row by row:

\[
\begin{align*}
v_{11} + 2v_{21} &= 2v_{11} \\
- v_{11} + 4v_{21} &= 2v_{21}
\end{align*}
\]

which are both equivalent to $v_{11} = 2v_{21}$. So, we may choose for $v_1$ any (non-zero) components that satisfy this relation, for instance $v_1^T = (2,1)$. Similarly, one finds for $\lambda_2 = 3$ the single condition $v_{21} = v_{22}$ and a possible eigenvector $v_2^T = (1,1)$. Now, we can return to the initial question that motivated our quest: "could we possibly transform $A$ into a diagonal matrix by a similarity transformation?" To see why we can for this choice of $A$, we only need to rewrite our defining condition for eigenvalues and eigenvectors $Av = \lambda v$ for the each of the two above possibilities using a single interesting trick: if we set $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ then the multiplication of any matrix $B$ by $\Lambda$ on the right has the effect of multiplying the first row of $B$ by $\lambda_1$ and the second row of $B$ by $\lambda_2$ (check it and generalize it to the $n \times n$ case). Let us try this trick for the matrix $B$ whose columns are $v_1$ and $v_2$. We find (check it!)

\[
AB = B\Lambda \tag{1}
\]

Now, it turns out that $B$ is nonsingular (we will see a general principle for this). So, we can multiply both sides of (1) by $B^{-1}$ to find

\[
B^{-1}AB = \Lambda \tag{2}
\]

which means that $A$ is similar to the diagonal matrix $\Lambda$ (or that $A$ is diagonalizable).

Let us now consider the case $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The characteristic polynomial reads $p(\lambda) = (1 - \lambda)^2 + 1$ with roots $\lambda = 1 \pm i$. We can no longer hope to find real eigenvectors since $Av$ would be real while $\lambda v$ would not! Indeed, we cannot even hope to diagonalize the matrix $A$ with real numbers since that would mean $B^{-1}AB = \Lambda$ or $AB = BA$ with $B$ and $\Lambda$ real and $\Lambda$ diagonal. But this would mean that the diagonal elements of $\Lambda$ would be eigenvalues and we cannot have any others than those already found since the characteristic polynomial of a $2 \times 2$ matrix is quadratic. However, we can diagonalize $A$ with complex $B$ and $\Lambda$. 
A third interesting case is given by \( A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \). The characteristic polynomial now reads \( p(\lambda) = (1 - \lambda)^2 \) with double root \( \lambda = 1 \). The trouble here is that the equation \( Av = v \) only has a one-dimensional family of solutions, those proportional to \( v^T = (0,1) \). So, the trick \( AB = BA \) requires that the two columns in \( B \) be proportional. Thus, \( B \) would be singular and one cannot diagonalize \( A \) in \( \mathbb{R} \) or \( \mathbb{C} \).

**Homework**

1. Find \( B^{-1} \) for \( B \) in (1) and verify (2) directly.

2. Diagonalize \( A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) with complex \( B \) and \( \Lambda \).