Linear Algebra (continued)

The fact that det(AB) $=$ (detA)(detB) has an interesting consequence in terms of similarity transformations. As we saw, if A is the matrix representation of the linear map T in the basis $\mathcal{E}$, and if B is the "change of basis" matrix from basis $\mathcal{E}$ to basis $\mathcal{U}$, then the map T is represented by the matrix $B^{-1}AB$ in the basis $\mathcal{U}$. How do the determinants of A and $B^{-1}AB$ compare? Using the above fact, we find that

$$\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B)$$

$$= \det(A)\det(B^{-1})\det(B) = \det(A)\det(B^{-1}B) = \det A$$

Since this holds for any change of basis, the determinant is common to *all* matrices that represent the same map T and can therefore be called the determinant of T itself.

Another important *invariant* of linear maps is the *trace*. Let us begin by a matrix definition. If A is an $n \times n$ square matrix of elements a_{ij} (in row i and column j), the trace of A is defined as the sum of its diagonal elements: $\text{traceA} = \sum_{i=1}^{n} a_{ii}$.

It is obvious that $\text{trace}(A + B) = \text{traceA} + \text{traceB}$. Slightly less obvious is that $\text{trace}(AB) = \text{trace}(BA)$. This results from writing the j^{th} diagonal element of AB as $\sum_{k=1}^{n} a_{jk} b_{kj}$ and summing over j to find

$$\text{trace}(AB) = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} b_{kj} = \sum_{k=1}^{n}\sum_{j=1}^{n} b_{kj} a_{jk} = \sum k^{\text{th}} \text{ diagonal element of BA}$$

Under a similarity transformation (viewed as a change of basis) we may thus write $\text{trace}(B^{-1}AB) = \text{trace}(ABB^{-1}) = \text{traceA}$. So, again, trace characterizes the linear map T, whatever matrix expresses it in whatever basis.

Eigenvalues

A diagonal matrix is the simplest one to deal with. One sees immediately the effect it has on any (column) vector: it simply scales each component according to the value of the corresponding diagonal element. It is trivial to see whether it is singular or not: one looks for a zero diagonal element.

So, the following question arises: could we possibly transform any $n \times n$ matrix into a diagonal one through a similarity transformation, that is by applying an adequate change of basis? The unfortunate answer is "not quite." First, there will be matrices that absolutely cannot be *diagonalized*. But they are rare. Second, among the large majority of those that can be diagonalized, the operation often requires the use of complex numbers, an embarassment when the vector space studied is real. However, the study of the question of diagonalization will yield great insight into the structure of linear maps. It begins with the notions of *eigenvalue* and *eigenvector*. The definition can be given for matrices or for the linear maps they represent in some basis. Just like the determinant and the trace, the eigenvalues are *invariant* with respect to the choice of basis. so, the definition can be given in abstract terms, with respect to the linear map T:

Definition: a scalar λ is called an eigenvalue of T if there exists a non-zero vector v such that $Tv = \lambda v$.

Of course, once a basis $\mathcal{E}$ is chosen for the n-dimensional space **V**, if A is the $n \times n$ matrix representation of T in that basis and x is the $n \times 1$ (column) matrix representation of v in that same basis, then we must have $\mathrm{A}\mathrm{x} = \lambda \mathrm{x}$. But this last equation can also be written $(\mathrm{A} - \lambda \mathrm{I})\mathrm{x} = 0$ which means that the matrix $(\mathrm{A} - \lambda \mathrm{I})$ is singular when λ is an eigenvalue. However, we have a test of singularity that will now take all of its importance: $\det(\mathrm{A} - \lambda \mathrm{I}) = 0$.

Let us investigate this condition with the matrix $\mathrm{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. The determinant of the matrix $(\mathrm{A} - \lambda \mathrm{I})$ reads with the standard notation

$$|\mathrm{A} - \lambda \mathrm{I}| = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2 = (\lambda - 2)(\lambda - 3) = \mathrm{p}(\lambda)$$

The expression $\mathrm{p}(\lambda)$ thus obtained is the *characteristic* polynomial of the matrix A (and of the linear map T it represents). The solutions λ of $\mathrm{p}(\lambda) = 0$ (here $\lambda = 2,3$) are the desired eigenvalues of A (or T).

Homework

1. (i) Assume that L is a lower triangular matrix with all diagonal elements 1. Show that its determinant is 1. Show that the same holds for an upper triangular matrix U. Hint: argue that L is the product of elementary matrices E_i corresponding to row additions. (ii) Now assume that A is non-singular. Prove that $\det(\mathrm{A}^T) = \det \mathrm{A}$. Hint, $\mathrm{A} = \mathrm{PLDU}$ with P a permutation matrix, L a lower triangular matrix with diagonal elements 1, D a diagonal matrix, and U an upper triangular matrix with diagonal elements 1.

2. Is it true that for any matrices A and B (of the same size):
(i) $\det(\mathrm{A} + \mathrm{B}) = \det \mathrm{A} + \det \mathrm{B}$, and that
(ii) $\mathrm{trace}(\mathrm{AB}) = (\mathrm{trace}\mathrm{A})(\mathrm{trace}\mathrm{B})$?