Linear Algebra (continued)

Let us return briefly to some ideas on basis and dimension in order to clarify a few points.

We defined a finite dimensional vector space **V** as one that has a finite spanning set and found that any basis of **V** must have the same number of vectors called the dimension of **V**. However, the field **F** over which we define our vector space can be **R** or **C** or even something else (such as the quarternions). Now, suppose we have a vector space over the complex numbers **C**. Since we know that **C** can be viewed as $\mathbf{R}^2$, the vector space **V** can also be viewed as one over the reals **R**. More precisely, suppose that we have a basis $\mathcal{E}=\{e_1,..,e_n\}$ of **V** viewed as a vector space over the complex. This means that any vector $v \in \mathbf{V}$ can be written uniquely $v = \mathrm{z}_1 e_1 + .. + \mathrm{z}_n e_n$ with $\mathrm{z}_j \in \mathbf{C}$. But since any z_j can be written uniquely $\mathrm{z}_j = \mathrm{x}_j + i\mathrm{y}_j$ (with x_j, $\mathrm{y}_j \in \mathbf{R}$ and $i^2 = -1$) we may write (uniquely)

$$v = \mathrm{x}_1 e_1 + \mathrm{y}_1 i e_1 + .. + \mathrm{x}_n e_n + \mathrm{y}_n i e_n$$

so that, if we let $f_j = ie_j$, the set $\mathcal{F} = \{e_1, f_1, ..., e_n, f_n\}$ clearly spans **V** viewed as a vector space over the reals **R**. Indeed, it is easy to verify that $\mathcal{F}$ is a basis for that space (see homework) and that it has $2n$ vectors. So, **V** viewed as a vector space on **R** has twice its dimension when viewed as a vector space on **C**. Some authors like to stress this fact by writing $\dim_{\mathbf{C}}\mathbf{V} = n$ and $\dim_{\mathbf{R}}\mathbf{V} = 2n$.

Another important issue is how one goes from one basis to another in a given vector space. Suppose that we have two bases $\mathcal{E}=\{e_1,..,e_n\}$ and $\mathcal{U} = \{u_1,..,u_n\}$ of **V** (now, without specifying over which field **F**). We may have a linear map T from **V** into itself currently represented by the matrix A in the basis $\mathcal{E}$. Presumably, a change of basis should imply some changes in the matrix that represents T. How does this work?

Consider an arbitrary vector $v \in \mathbf{V}$. It can be written

$$v = \mathrm{x}_1 e_1 + .. + \mathrm{x}_n e_n$$

or (1)

$$v = \mathrm{y}_1 u_1 + .. + \mathrm{y}_n u_n.$$

Moreover, the vectors u_i can be seen as the images of the e_i's by some linear map S from **V** into itself as follows (see homework):

$$\forall i, \text{ let } u_i = Se_i \text{ and}$$

$$\text{if } v = \mathrm{x}_1 e_1 + .. + \mathrm{x}_n e_n \text{ then } Sv = \mathrm{x}_1 u_1 + .. + \mathrm{x}_n u_n \qquad (2)$$

Let us call B the matrix that represents S in the basis $\mathcal{E}$ (it has the components of u_j as its jth column). Writing the second line of (1) in the basis $\mathcal{E}$ then yields the equality x = By

So, if we have a vector v written in the basis $\mathcal{U}$ as a column matrix y, we obtain easily its representation x in the basis $\mathcal{E}$ by x = By. Conversely, since B is clearly nonsingular, we obtain y from x by: y = B^{-1}x.

So, if we know the matrix A of T in the basis $\mathcal{E}$, we know how to obtain in $\mathcal{E}$ the components of Tv (the column matrix Ax) from those of v (the column matrix x). If we now want Tv in $\mathcal{U}$ from y in $\mathcal{U}$ we can simply apply B^{-1} to Ax and rewrite x = By to obtain the column matrix in $\mathcal{U}$ of Tv as: $B^{-1}ABy$. This means that the matrix M that represents T in the basis $\mathcal{U}$ is M = $B^{-1}AB$. We say that M and A are similar and that the transformation A $\rightarrow$ $B^{-1}AB$ is a similarity transformation. The "being similar" relation is easily shown to be an equivalence relation (see homework) among $n \times n$ matrices. A particular case of interest is when B is orthogonal. In that case A and M are called "orthogonally equivalent."

Similarity transformations will be particularly useful when we will try to rewrite a given matrix in a more practical form. For instance, we will show that a (real) symmetric matrix is orthogonally equivalent to a real diagonal matrix.

Homework

1. Show that the set $\mathcal{F} = \{e_1, f_1, ..., e_n, f_n\}$ with $f_j = ie_j$ is linearly independent in the real if the set $\mathcal{E}$=$\{e_1,..,e_n\}$ is linearly independent in the complex.

2. Verify that S defined by (2) is a linear map.

3. An equivalence relation among elements of a set $\mathcal{S}$ is a relation $s \leftrightarrow t$ satisfying for any s, t, u:
(1) $s \leftrightarrow s$; (2) $\{s \leftrightarrow t\} \Rightarrow \{t \leftrightarrow s\}$; and (3) $\{s \leftrightarrow t$ and $t \leftrightarrow u\} \Rightarrow \{s \leftrightarrow u\}$
Show that the similarity transformation between matrices defines an equivalence relation.