Introduction

Welcome to Math 500 "Markov Processes, Decisions, and Evolution." Seminars, such as this one, are often difficult to describe adequately in a few words, especially when they involve several interconnecting aspects of mathematics and their applications. This course, for instance, could have been equally well titled "Matrices: Topics and Applications" with the study of eigenvalues, in particular those of random matrices as topics, and Markov chains, dynamical systems, and evolutionary game theory as applications.

This seminar is the first in a series of six semesters under a collaboration between the Mathematics Department at San Francisco State University and the Mathematical Sciences Research Institute (MSRI) in Berkeley, funded by the National Science Foundation. The purpose of the collaboration is to give access to the programs organized by MSRI to our students and faculty. During the Spring 1999 Semester, MSRI will be running a program on Random Matrix Methods and this seminar will devote several weeks to an introduction to that topic. Speakers from MSRI will be invited to give talks to the class and students in the class will attend one talk at MSRI.

The subject of random matrices is very young (by mathematical standards) and highly specialized. It arises from various problems in physics and finds some surprising connections with other branches of mathematics such as the study of the zeros of the Riemann Zeta function. In the study of random matrices, one focuses on special sets of matrices (Gaussian ensembles) with useful properties (Hermitian, symmetric, unitary) and derives probability laws for the distributions of their eigenvalues.

The subject of Markov chains (or processes) is a much older one, although not that old by mathematical standards. It is concerned with the stochastic evolution of a system when the laws of probability that govern it are such that the past doesn't matter. Markov chains have numerous uses in science and engineering, for instance in population dynamics, in nuclear physics, and in the management of telephone networks. The study of stochastic (transition) matrices is fundamental to the analysis of Markov chains.

Dynamical systems, whether stochastic or deterministic, appear prominently in the applications of mathematics. They involve some of the most arduous mathematical problems. The study of the dynamic stability of an equilibrium makes use of the eigenvalues of a (Jacobian) matrix associated to the laws that govern the system.

Evolutionary game theory is also a very young field that grew out of the slightly older field of game theory from an interaction of ideas between biologists, economists, and political scientists. If individuals in a society interact in the pursuit of their own self interest, what kind of social outcome can be expected, especially in circumstances when excessive competition can deteriorate the environment or impair the fitness of the population? The finding that cooperation can spontaneously and rationally evolve in a society of egoists has been striking but still lacks somewhat in explanations. Techniques from game theory, Markov chains, and dynamical systems will help clarify some of the issues.

The seminar will touch on all of these subjects and will attempt to develop their interconnections. The primary goal will be to expose the students to the process of pure and applied mathematical research and motivate them to pursue it further. In this spirit, students will be expected to complete an individual research project on a topic related to the course. This can involve a literature search, simulations, computation, or the proof of some results. Weekly homework assignments will be given, collected, and graded. Course grade will be based on 30% homework + 20% class participation + 50% project. Class notes will be posted regularly on the instructors' webpage at:

http://userwww.sfsu.edu/~langlois/teaching

Outline

One prerequisite for this seminar is a first course in linear algebra (Math 325 at SFSU). In such a course, one usually begins with linear equations, Gaussian elimination, LU decomposition, then goes on to linear independence, inner product and orthogonality, and finally introduces determinants and eigenvalues. A second course would be a bit more abstract and would give a more thorough treatment of eigenvalues and eigenvectors especially as they relate to important classes of matrices. The first few class meetings will be devoted to a quick review and further developments of linear algebra. We will look (among other things) at Hermitian and unitary matrices since a great deal can be said about their eigenvalues (real and on the unit circle respectively). In turn, these are classes where a great deal can be said in the random case.

In the second part of the course, we will study simple Markov chains, first in discrete time, and then in continuous time. We will look at the notion of stationary distribution, expected return time, and the ergodic theorem that describes the long term properties of Markov chains. Eigenvalues of the associated transition matrices are useful and interesting in themselves.

In a third part, we will conduct a brief study of dynamical systems and of the relation between dynamic stability of an equilibrium and the eigenvalues of the associated Jacobian matrix.

In the fourth and last part, we will study elements of evolutionary game theory and consider some applications of Markov chains and dynamical systems to that subject. One case of great interest is the famous Prisoner's Dilemma problem. The story goes as follows: two prisoners suspected of committing a crime are placed in separate cells and offered the following bargain: Prisoner A (say) may confess his crime and thus turns in evidence against his accomplice B. B is of course offered the same bargain to the full knowledge of both. If both suspects confess, they will receive a quite heavy sentence. If neither confesses, they will likely be convicted on lesser charges. But if A alone confesses, he will be treated very leniently while B will get the book thrown at him (and conversely if B alone confesses).

Each prisoner finds that, no matter what his complice does, he is better off confessing (neglecting the possibility of later revenge). So, the logical outcome of this game is that both prisoners confess. The paradox is that they would both be better off if neither confessed. Many problems arising from economics, sociology, political science, and population biology, have this prisoner's dilemma-like structure. Arms races and economic competition between two firms are typical examples. The only difference is that the decision problem is repeated in time and the "players" can base their decisions on the past. Theoretical and empirical results support the finding that players often evolve strategies that allow them to maintain cooperation by relying on various punishments for defection. However, there are numerous such strategies and the question remains open as to how they are found and adopted by a majority of the population.

Questions arising from all these topics and related ones provide many attractive subjects for student projects. The students are strongly encouraged to talk to the instructor early on about choosing a topic.

A Brief Review of Basic Linear Algebra

Linear algebra begins with the problem of finding the solutions $(x_1, x_2, .., x_n)$, given coefficients a_{ij} (i=1,..,m, j=1,..,n) and b_i (i=1,..,m), of the m linear equations

$$a_{i1}x_1 + .. + a_{ij}x_j + ... + a_{in}x_n = b_i \tag{1}$$

which is also written

$$Ax=b \tag{2}$$

with the "matrix" $A = \begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ and the "column vectors" $x = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \cdots \\ b_m \end{pmatrix}$, and the corresponding definition of the matrix multiplication of A by x. At first, the elements involved in A, x, and b, are assumed real, but this can be extended to the complex numbers.

A case of great interest (that we will concentrate on from now on) occurs when $m = n$ since very concise results can be stated. They rely on the notion of "singular" and "nonsingular" matrices. To obtain the result, one first define "row operations" as any combination of the following three: interchange of two rows in (1); multiplication of one row by a non-zero scalar; addition of a row to another. The process is often called "Gaussian elimination."

It is easily seen that such operations do not change the nature of the solutions (if any) and that they amount to the multiplication of A (and b) by so-called "elementary matrices" E_k that represent each of the actions. Moreover, they can be used systematically to change the system of equations (1) into a more tractable one by eliminating the terms "below the diagonal." Indeed, the operations can be carried out directly on A without regard to the specific b that defines the equation. For example, let

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & -3 \\ -2 & 3 & 3 \end{pmatrix} \tag{3}$$

Adding row 1 to row 3 is the same as (left) multiplying A by $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ to obtain $E_1A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & -3 \\ 0 & 6 & 3 \end{pmatrix}$. Now, subtracting twice row 2 from row 3 is the same as multiplying E_1A by $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ to obtain $E_2E_1A = U = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 9 \end{pmatrix}$. Clearly, the equation $Ax = b$ is equivalent to the equation $E_2E_1Ax = E_2E_1b$ which is far easier to solve since one can start solving the third line for x_3, then use that in the second line to get x_2, and then use both in the first line to obtain x_1.

One great advantage of row operations and the elementary matrices they involve is that they are fully reversible. Subtracting row 1 from row 3 and adding twice row 2 to row 3 will bring U back to A. The corresponding elementary matrices are just like E_1 and E_2 except that the off-diagonal 1 in E_1 is replaced by -1 and the -2 in E_2 is replaced by $+2$. If we call these matrices respectively E_1^{-1} and E_2^{-1}, one finds that $E_1^{-1}E_1 = I$ and $E_2^{-1}E_2 = I$ where I is the "identity" matrix with ones on its diagonal and zeros elsewhere. One also verifies easily that $L = E_2^{-1}E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$ and that $A = LU$. This is the famous LU decomposition of A into the product of an "upper" triangular matrix U by a "lower" triangular matrix L.

Can anything go wrong? Not much. One minor hitch could happen, for instance, with $A' = \begin{pmatrix} 0 & 3 & -3 \\ 2 & 3 & 0 \\ -2 & 3 & 3 \end{pmatrix}$. We would need to first exchange (say) rows 1 and 2. This would be the same as multiply A' by $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to produce $A = PA'$ with the same further steps as above. P is easily seen to be its own inverse $P^{-1} = P$. So, reversing that step is easy. But A' cannot be directly written into the form LU. Instead, it is so up to the "permutation" P and we have PA'=LU. In general, several permutations could be involved in P (in which case the inverse of P involves its elementary constituents in reverse order).

Interestingly, all this could be done whether or not the final diagonal in U is made up of non-zero scalars. But if it does have a zero scalar, the solution of Ax=b is not guaranteed to exist and the matrix A is called "singular." If all diagonal elements in U are non-zero, the solution is not only guaranteed to exist, it is also unique and the matrix A is "nonsingular."

In the nonsingular. case, the row operations can be carried out further with interesting results. All diagonal elements can be scaled to take the value 1. For U, this is achieved through multiplication by the "elementary matrix" S $= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}$ which is obtained from the "diagonal" matrix D $= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ by taking the inverses of the diagonal elements. Clearly, DS $=$ I and we can write PA' $=$ LDU' where U' $=$ SU is clearly upper triangular.

We will refer to this principle as the "LDU" decomposition. It is very useful in the theory of determinants (depending on how it is developed) because L and U are triangular with ones on their diagonals while D is diagonal only. Indeed, up to the "sign" of the permutation defined by P (whether it is odd or even) the determinant will be just the product of the diagonal elements of D.

Furthermore, all elements above the diagonal in SU can be eliminated by row operations (multiplying by elementary matrices) to yield the identity matrix I. In summary, we have the following characterization (theorem): a matrix is nonsingular if and only if it is row equivalent to the identity matrix. By anticipation, it is clear that it is so if and only if its determinant is non-zero.

Homework

1. Rewrite A $= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ in the from A $=$ P^{-1}LDU (give P, L, D, and U).

2. Assume that A is such that A $=$ LU where L is lower triangular with ones on the diagonal (and zeros above the diagonal), and U is upper triangular with non-zero elements on the diagonal (and zeros below the diagonal). Argue directly (do not use other theorems of linear algebra) that Ax $=$ b has exactly one solution x given b. Hint: let y $=$ Ux and argue that Ly=b has exactly one solution.

3. The transpose A^T of the square matrix A $= (a_{ij})$ $(i,j = 1,..,n)$ is the matrix (a_{ji}) obtained by exchanging rows and columns. Assuming (you may wish to prove it) that $(AB)^T = B^T A^T$ argue directly and carefully from the LDU decomposition (nothing else) that A is non-singular if and only if A^T is non-singular.