Chapter Three: Non-Repeated Complete Information Games*

3.1 Basic Game Modeling

Game-theoretic modeling often begins with the simplest of structures, either in extensive or in normal form. Such basic structures are meant to define the players, their available actions and their priorities. The basic structures can then be extended in two main directions: incomplete information or their repetition with discounting of the future.[1] However, even basic game structures can provide meaningful and relevant models of real-world issues.

3.2 Normal Form Games

As was seen in Chapter One, the normal form of a game does not adequately capture issues of timing or information. But when such issues are not central the normal form provides an attractive modeling option. This is often the case when two or more individuals make quasi-simultaneous strategic decisions that cannot be revisited in view of the unfolding of the game and that involve no expectation of consequences in future game playing. The Prisoner's dilemma and the Battle of the Sexes, at least in the terms discussed previously, are typical examples: the two prisoners make their decisions independently while possible future retaliation is neglected. Indeed, introducing the very possibility of retaliation profoundly affects the strategic picture, a subject that will be discussed at length in Chapter Five. Similarly, the Battle of the Sexes is viewed as a one-shot issue: perhaps the couple failed to communicate beforehand and must make their respective decisions on the spot without the chance to consult each other. Better communication could solve the problem by providing an expectation of what the other will do. However, the strategic issue remains whole when they attempt to reach an expectation: indeed, there are three Nash equilibria in that game. So, which one will prevail?

3.2.1 Some Famous Basic Games

The Battle of the Sexes belongs to a class of "coordination games." The game has two pure equilibria: He chooses the fight and She goes along. Or She chooses the ballet and He goes along. There is a third Nash equilibrium where they each make probabilistic choices that depend on how much they value their respective outcomes. That solution is picture in Figure 3.1.

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[1] It is possible to study repeated games with incomplete information but these are very advanced topics beyond the scope of these lecture notes.

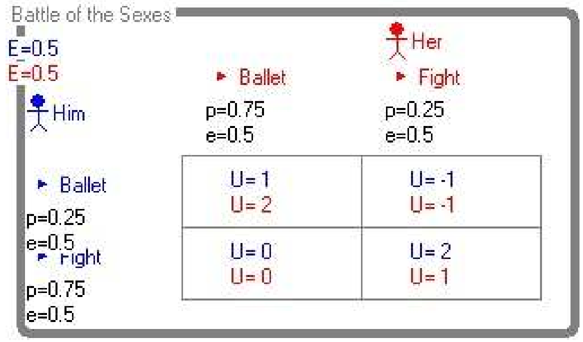

Figure 3.1: A Solution of the Battle of the Sexes

It is easy to verify that the probabilities involved change with the respective magnitudes of the payoffs each side assigns to the various outcomes: one simply edits the game in GamePlan and solves it. Coordination games can be far more complex than that. They can involve more choices and more players. Another famous such game is the "Stag Hunt" (see homework.)

Another prominent normal form game is Chicken. This is supposedly a formal model of the "game of Chicken" described by Herman Kahn as a metaphor for nuclear crises.[2] The game, supposedly played by California teenagers in the 1950s, involves two cars speeding toward each other in the middle of the road. The first one to swerve is "chicken." The game is really one of timing and uncertainty about the other side's resolve. But game theory was not that sophisticated at the time and the "game model" pictured in Figure 3.2 was proposed.

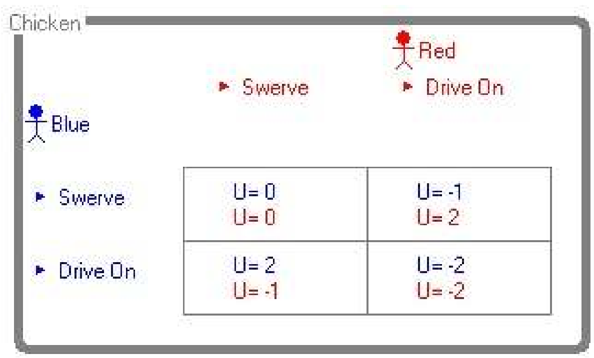

Figure 3.2: Chicken

If each side commits in advance to a strategy it has to make assumptions about the other. The standard search for a stable cell, in the best reply to best reply process, can, for instance, begin in the up-right one: under that assumption Blue would not choose to Drive On and Red would not choose to Swerve. So, that cell provides a Nash equilibrium. But so does the

[2] "Thinking about the Unthinkable" (1962.)

down-left cell where Blue drives on and Red swerves. Worse, there is also a mixed-strategy equilibrium that begs for an interpretation.

Volumes have been written about the above game of Chicken, mostly in the International Relations literature where it is still widely viewed as a game model of nuclear crises. Needless to say, volumes have also been devoted to criticizing it as a model and improving it in various ways.[3]

3.2.2 More Strategies and More Players

There are several interesting structures that arise from the so-called 2 by 2 games, meaning normal form games with just two players and two strategies per player (see homework.) But the possibilities explode geometrically when adding strategies and players. A two-person game with three strategies per player (3 by 3) is pictured in Figure 3.3.

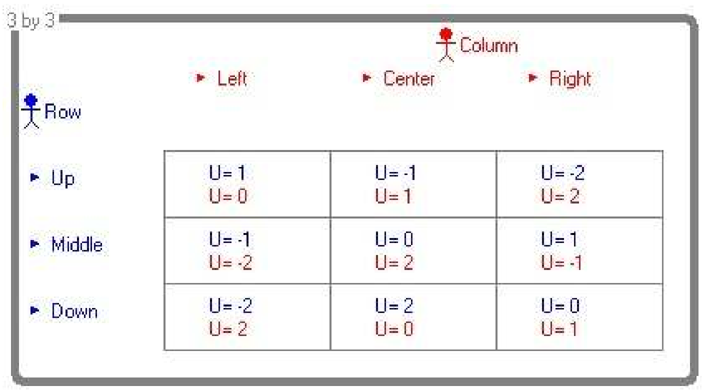

Figure 3.3: A “3 by 3” Game

The standard search for a stable cell can begin, for instance, in the up-left corner. Row would not want to switch strategy from there but Column would switch to Right, followed by Row’s switch to Middle, Column’s switch to Center, Row’s switch to Down, Column’s switch to Left and Row’s switch to Up, back to the beginning. There is, indeed, no pure strategy Nash equilibrium in that game. There is, however, a single mixed-strategy equilibrium.

A three-person game with two strategies per player is pictured in Figure 3.4 using the GamePlan format. The left-hand side table shows the payoffs when Green chooses Coop while the right-hand side table shows them when Green picks Dfct. This is in fact a possible version of a 3-player Prisoner’s Dilemma with a twist: if all three prisoners confess (Dfct), the evidence becomes overwhelming and the prosecutor sends them all off to prison for life! As a result, the

[3] This author’s modest contribution to that debate is “Rational Deterrence and Crisis Stability” in the *American Journal of Political Science* (1991.)

unanimous choice of Dfct does not form an equilibrium and one side should instead hold off confession (Coop) expecting the other two to tell on each other. But this yields three possible pure-strategy equilibria, one for each of the three possible sides that chooses to not confess. There are in fact four further mixed-strategy equilibria (see homework.)

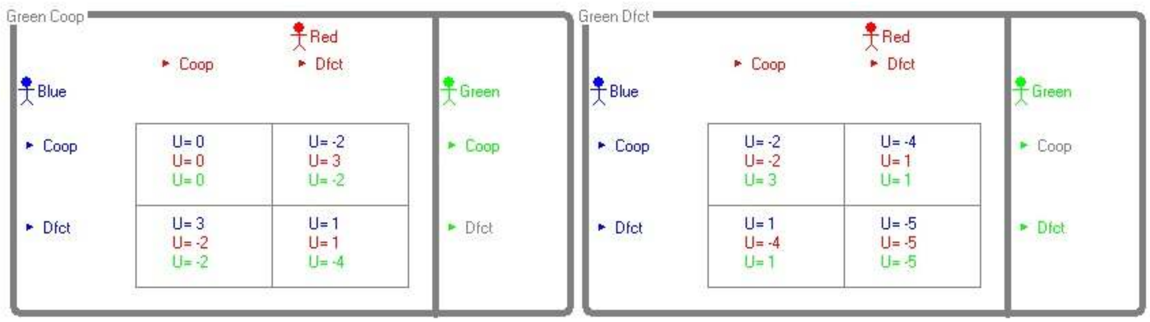

Figure 3.4: A “2 by 2 by 2” Game

Three or more player games can offer interesting modeling opportunities, especially to understand how some subset of players can coordinate their play successfully to the expense of the others.

3.2.3 Zero-Sum Games

John von Neumann was first to establish that normal form two-person zero-sum games always admit an equilibrium. Such a game is easy to spot: the payoffs in each cell must add up to zero (or to any given constant.) One easily sees that the game of Figure 3.3 does not have zero-sum. One game that has zero-sum is pictured in Figure 3.5.

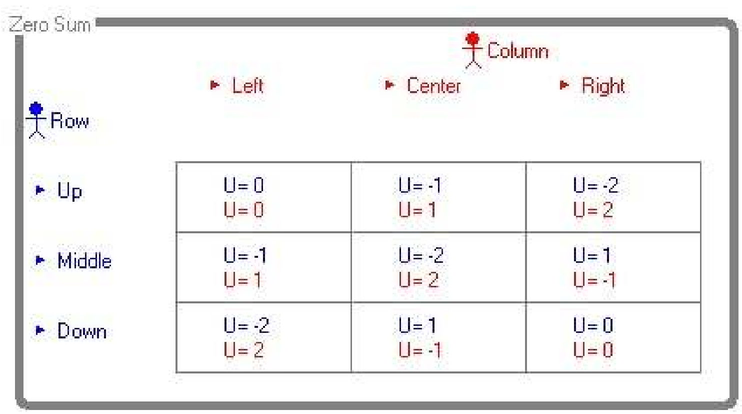

Figure 3.5: A Zero-Sum Game

The same kind of search for equilibrium, a stable cell, can be conducted, sometime successfully if a pure-strategy equilibrium does exist. Here, it is easy to see that this is not the case. The only difference between zero and non-zero sum normal form games really lies in the characterization of their solution set: in zero sum games, any convex combination of solutions is still a solution whereas this doesn't hold in non-zero sum games. So, the solution of a zero sum game is always a convex set.

An interesting class of zero-sum games is given by issues of optimal allocation of resources between several theaters of conflict. Say there are three theaters and anyone who allocates more assets than the other on one theater wins that battle. If they allocate the same number of assets to that theater, it is a draw. The objective, of course, is to win the war. This class of games is known as "Colonel Blotto" games (see homework.)

3.2.4 Solving Normal Form Games

Aside from identifying dominant strategies, the simplest approach to solving a normal form game is the search of a stable cell in the best reply to best reply process that has been successfully applied to a few games so far. Unfortunately, that search will succeed only with a very small subset of games, those that have a pure-strategy equilibrium. For the vast majority of other games there is a need for more systematic and efficient techniques.

It is not difficult to show that a two-person zero-sum game is equivalent to a linear programming problem that has a solution. Indeed, John von Neumann's result is often called the "Minimax Theorem" because it shows the existence of a value V for the game, the optimal payoff received by one side that attempts to maximize it while the other (which receives –V since the game is zero-sum) attempts to minimize it. Linear programming problems can be solved systematically with various techniques of optimization. A standard technique is the "Simplex Algorithm."

A two-person non-zero sum game in normal form is much harder to solve. The main technique is called complementary pivot programming, more specifically the Lemke-Howson Algorithm. [4] Contrary to zero-sum games, that method is not guaranteed to identify all equilibria and its extension to more than two players is just as unreliable. Various studies have established that solving non-zero sum games in normal form, especially those with more than two players, are "hard" problems that involve high mathematical complexity.

3.3 Extensive Form Games

The extensive form is the best modeling option when timing or information are important aspects of the players' strategic thinking. In particular, if players can adapt their choices to what is observed in the unfolding of the game, then chances are that the normal form could erase some critical strategic aspects.

[4] "Computing Equilibria for Two-Person Games," Chapter 45, *Handbook of Game Theory with Economic Applications*, Vol. 3 (2002), eds. R. J. Aumann and S. Hart, Elsevier, Amsterdam, pages 1723–1759.

3.3.1 Open versus Secret Ballots

A panel of three judges must vote on an appeal. Bob, the Chief Judge is a hard liner. He wants to uphold the law and tends to resent all appeals for leniency based of flimsy mitigating factors, as in the case under review. Gina is a liberal but also a consensus builder. In her heart of hearts she would like to grant the appeal, but she values highly her colleagues' opinions. She values a unanimous vote to reject the appeal as highly as seeing the appeal granted by a vote of two to one. Ronald is all for toughness when it comes to crime, but he is a born dissenter. And dissenting with the chief judge is what motivates Ron the most. His favorite outcome is a vote of two against one, with Bob casting the dissenting vote. His next best is to be in the minority.

Bob's as chief judge must decide on a voting procedure. One possibility is to ask each of his colleagues in turn, according to seniority, to express their vote to the panel. The meeting would begin with Bob announcing his vote rejecting the appeal. He would then ask Gina and Ronald to announce their preference in turn. In order to know what to expect, Bob sets up the Sequential Voting game model of Figure 3.6.

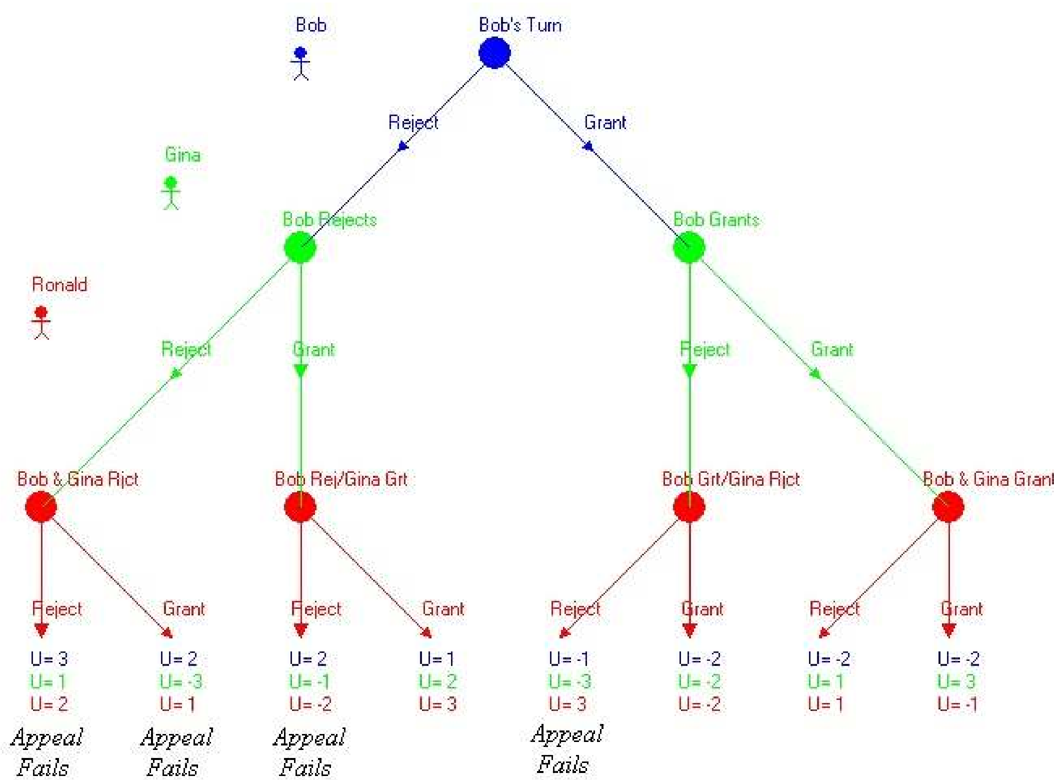

Figure 3.6: Sequential Voting

Bob having perfect knowledge of the game chose each judge's payoffs carefully to reflect their known priorities. He also knows how to apply the backward induction of rational behavior from the end moves. But he has an even more effective solving tool: a copy of GamePlan that he received as a gift from an admiring clerk. The solution of the game comes as a reality check: Ronald and Gina vote against him to grant the appeal!

Perhaps changing the voting order would produce a different result? But this would create protocol issues: Gina could object on being last to vote, after the two "boys." So, Bob has an idea: he can hold a secret ballot vote! He edits his GamePlan file and solves the game to obtain, to his delight, a completely rational unanimous vote to reject the appeal.

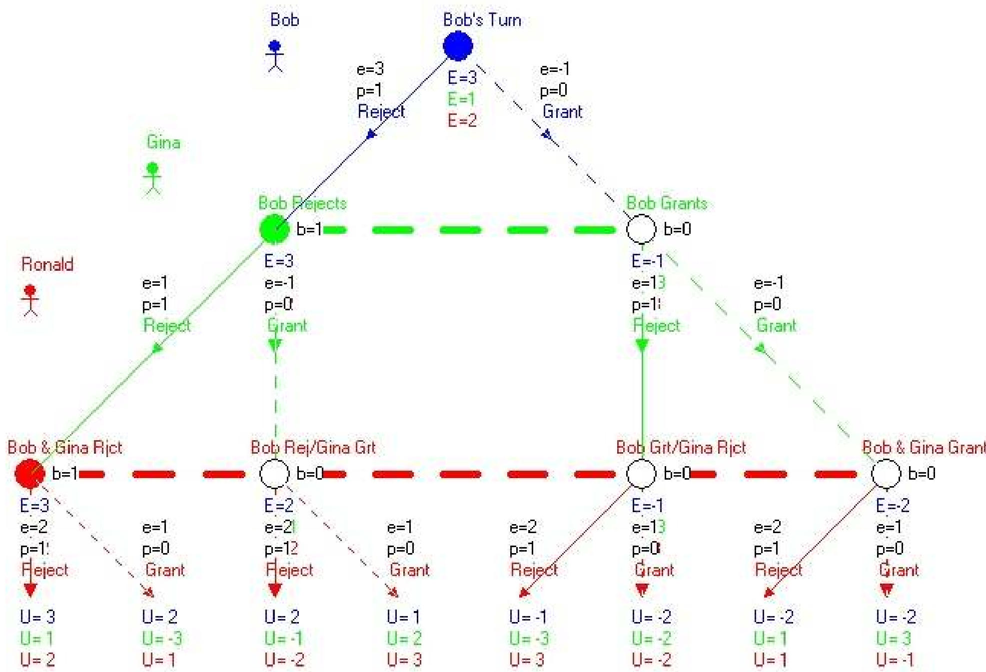

Figure 3.7: A Solution of the Secret Ballot Vote

Unfortunately, the same solution as in the sequential vote also appears as well as a probabilistic one (see homework) according to GamePlan. But, at least, Bob now has a shot at his favorite outcome by adopting a perfectly legal voting procedure!

3.3.2 A Strategy for the Prom

Gwen is a senior at her high school and she is admired by many of her classmates. Indeed, Gwen is smart and pretty, and she is one of the best athletes in the school. Yet, Gwen's unusual mix of qualities may be a serious handicap in the dating game for the forthcoming Prom night. Indeed, she so impresses her peers that none of her classmates has managed to foster the courage to ask her to be their date at the prom.

Gwen's best friend Linda is well aware of this paradoxical situation. So she decides to take action. She leaks to the class that Gwen once confided in her that she really likes a tall guy with green eyes and a talent for chess. Unfortunately, there are two boys in the class who fit that profile very well: Bill and Rob. Needless to say, both are crazy about Gwen. Unfortunately, they are also very sensitive guys and thus extremely afraid of being rejected.

Gwen in fact likes Bill and Rob equally. But she is a bit vain and would be most flattered if both boys decided to ask her, so that she could have her pick. In order to help sort things out, Linda (who has studied Game Theory online) set up the model of Figure 3.8:

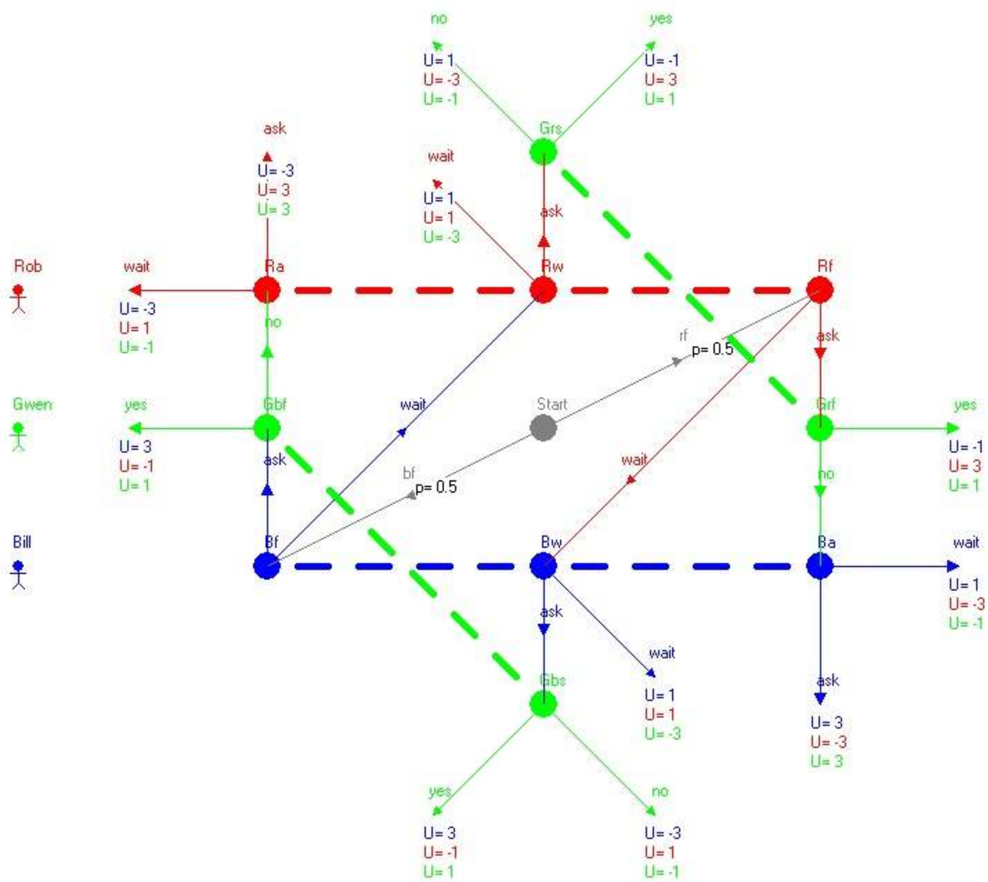

Figure 3.8: A Model of the Dating Game

In Linda's thinking, Nature is going to choose randomly (and fairly) one the two boys to make the first decision. But when he makes his decision, each boy is uncertain about whether:

a. He is the first mover;
b. The other boy just chickened out; or
c. The other boy asked Gwen but she turned him down.

And at her turn Gwen only knows which of the two boys is asking her. She doesn't know whether he is the first mover or if the other boy has already been scared away. Gwen likes certainties. She doesn't want a strategy that involves flipping a coin. So, Linda explains that a pure-strategy equilibrium of her game model would provide the most rational plan for Gwen under the circumstances. Unfortunately, there are two such equilibria with a rather unappealing property: in each solution, Gwen should decide to always reject one of the boys and always say yes to the other. The only difference is who that is!

Linda did try to convince Gwen to look at probabilistic solution as well. But the fact is that they don't provide better value. In fact, they do not improve at all on the expected payoff for Gwen at the beginning of the game (at the Start node.) In the end, Linda had an idea: she asked Gwen whether she had even a tiny preference for either boy? Gwen did confess a slight preference for Bill. Game Theory finally had an answer: don't be vain, just decide to say yes to Bill and no to Rob, no matter what (see homework.) Of course, this has to assume that Bill will not chicken out…

3.4 Continuous Games

A game is called "continuous" when the choice set of at least one player is the continuum to begin with, not just an extension of a discrete set to the probability distributions over that set. It may merely be a matter of interpretation when payoffs are linear in the choices. But it is unambiguous when payoffs are non-linear. Such non-linearities arise quite naturally in Economics. The Cournot Duopoly and Oligopoly models of Chapter One are good examples. Another good example is the Bertrand Duopoly.

3.4.1 The Bertrand duopoly

Joseph Bertrand, in a critique of Augustin Cournot's duopoly model in the late 19th Century, pointed out that most businesses choose price rather than quantity. The idea that businesses compete on price can be approached in several ways. The following model is a relatively simple one.

Assume that the consumers for a homogeneous product (such as gas) are distributed uniformly along Main Street, pictured as the segment [0, 1] of the x-axis. Two suppliers A and B are located at points a and b on that segment. We assume $0<a<b<1$. Business A charges price p_A and business B charges price p_B for the exact same product (say, a gallon of gas.)

Typical consumer x (located at position x) derives utility $U_A=U-p_A-c(x-a)^2$ by shopping at A and $U_B=U-p_B-c(x-b)^2$ by shopping at B, where $U>0$ is his satisfaction of obtaining the product. The quadratic term is a cost of traveling to either place. Clearly, consumer x will prefer shopping at A when $U_A< U_B$. Solving for equality $U_A= U_B$ yields the critical consumer z such that all $x<z$ shop at A and all $x>z$ shop at B. One has:

$$z = h+(p_B- p_A)/2cd \qquad (1)$$

with $d=b-a$ and $h = (a+b)/2$. So, with the given prices, business A will capture z customer and business B will get the remainder (1-z). If we further assume the same linear costs of supply kz for quantity z for both, we can write the profit (objective) functions for each business:

$V_A = z\ (p_A\text{-}k)$ and $V_B = (1\text{-}z)\ (p_B\text{-}k)$ (2)

In order to optimize profits, one differentiates V_A with respect to p_A to obtain the optimum

$p_A = cdh + ½\ (k + p_B)$ (3)

and, similarly (verifying down concavity of profits)

$p_B = cd(1\text{-}h) + ½\ (k + p_A)$ (4)

As with Cournot, an equilibrium is achieved by solving (3) and (4) simultaneously, yielding:

$p_A = k + (2/3)cd(1+h)$ and $p_B = k + (2/3)cd(2\text{-}h)$ (5)

There are many variations that are Bertrand-type duopolies that make proce rather than profit appear as the decision variable (see homework.)

3.4.2 A General Theorem

There are several possible generalizations of Nash's Theorem to various types of games. The standard requirements for continuous games are:

a. Continuity of payoffs in all decision variables;
b. Convexity and compactness (the quality of being closed and bounded) of each player's decision space;
c. Concavity, or at least "quasi-concavity" of the players' own payoffs in their own decision variables.

Under such conditions there always exists an equilibrium.

3.5 Homework

3.5.1 Rock-Paper-Scissors.

Write the payoff matrix of this well known game and obtain the single Nash equilibrium.

3.5.2 The Stag Hunt

The original story is attributed to jean-Jacques Rousseau: two men can go hunting for rabbits individually, or they can join forces and hunt for a stag. Write a simple normal form representation of this typical social dilemma between individualism and cooperation and comment on your solutions. Generalize to a three player case where only two need to join forces to succeed in their hunt.

3.5.3 The Tragedy of the Commons

When more than two players are involved in a Prisoner's Dilemma like situation, one often refers to the so-called "Tragedy of the Commons," the idea that individualistic behavior can deplete common resources to the detriment of all, if left unchecked. Generalize the Prisoner's Dilemma of Chapter One to a 3-player symmetric game. Discuss the following two cases on how you attribute payoffs according to the number of defections: (1) when it is best to defect against two defectors; and (2) when it is worst to defect against two defectors. Solve for Nash equilibria in each case and comment.

3.5.4 Reverse orders in voting

Exchange the order of play for Gina and Ronald in the sequential game, solve and comment.

3.5.5 The Dating Game

Solve the Dating game of Figure 3.8 for pure equilibria. Then adjust Gwen's payoff to reflect a slight preference for Bill and justify Linda's conclusions.

3.5.6 The Bertrand Duopoly

Redo the solving done in the text, replacing cost kz by kz^2 in the payoff functions of the two firms.