Does the Principle of Convergence Really Hold?

War, Uncertainty, and the Failure of Bargaining

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Abstract

Convergence occurs in war and bargaining models as uninformed rivals discover their opponent's type by fighting and making calibrated offers that only the weaker party would accept. Fighting ends with the compromise that reveals the other side's type. We show that, if the protagonists are free to fight and bargain in the time continuum, they no longer make increasing concessions in an attempt to end the war promptly and on fair terms. Instead, our rivals stand firm on extreme bargaining positions, fighting it out in the hope that the other side will give in, until much of the war has been fought. Despite ongoing resolution of uncertainty by virtue

*The order of the authors' names is not indicative of their respective contributions that they consider equal. We thank anonymous reviewers for extensive and insightful feedback.

of time passing, the rivals choose not to try to narrow their differences by negotiating.

Introduction

Reflecting upon issues of bargaining in war, James Fearon points to the 'nearly universal tendency for states at war and parties to civil conflicts to simply fight for extended periods of time without making serious offers for a negotiated settlement.'[1] Paul Pillar, in a detailed examination of bargaining behavior between states at war, confirms that warring states rarely attempt to negotiate.[2] Instead he finds that the parties negotiated *before* an armistice in only nineteen of one hundred and forty two interstate wars examined. And even when states engaged in pre-armistice negotiations, 'it was only after a period of fighting without talking that the parties began to talk..'.[3] The Indochina war had lasted for eight years before France and the Viet Minh entertained a 'first open and direct contact' over the exchange of prisoners in July 1954.[4] At the close of the Russo-Japanese war of 1904, Japan and Russia finally agreed to talk at Portsmouth in July 1905 having fought without talking for over a year. As John White reports, by early 1905 'the willingness of the Russian government to seek peace was as halfhearted as its desire to avoid war had appeared.'[5] Clearly, states fight *without* negotiating for much of the duration of a conflict. Yet states learn from the battles that they fight. Japan understood as early

[1] Fearon, 'Fighting Rather than Bargaining,' (unpublished manuscript, Stanford University, 2007) p. 26.

[2] Paul R. Pillar, *Negotiating Peace. War Termination as a Bargaining Process*, (Princeton University Press, Princeton, New Jersey, 1986).

[3] Pillar, *Negotiating Peace*, p. 44.

[4] Edgar O'Ballance, *The Indo-China War, 1945-1954*, (Faber and Faber, London, 1964) pp. 245-246 as quoted in Pillar, Negotiating Peace, p. 79.

[5] John A White, *The Diplomacy of the Russo Japanese War*, (Princeton University Press, Princeton New Jersey, 1964) p. 206.

as June of 1904 that 'the general image of her as the weaker party was, in the fundamental sense, correct.'[6] So why did the warring parties remain silent? Can rationality explain such behavior?

In the absence of commitment issues, explaining silence in war requires understanding why states at war would choose to fight without probing the other side for a possibly war-ending settlement. But such probing is at the core of the Principle of Convergence according to which wars should end on mutually agreeable terms when uncertainty about the other side has been resolved, and fighting no longer serves to inform on the terms of a settlement. War and bargaining models describe the process that underlies this principle by specifying the screening procedures that leads to the identification of the players' true prospects through bargaining and the observation of battle outcomes.[7] The process is designed to separate player *types* so that the rivals can avoid as much costly fighting as possible while making peace without conceding more than is necessary for the opponent to agree to settle. It requires that rivals bargain while they fight, adjusting their offers as they observe the outcome of battle and the other side's negotiating behavior, and it precludes fighting to pressure the other side to surrender to outstanding demands. While this process is formally rational and efficient, it remains at odds with what we know about states at war.

If we are to explain silence in war as a rational choice, barring commitment

[6] White, *The Diplomacy of the Russo Japanese War*, p. 199.

[7] Technically, uncertainty is reduced by a screening process that separates the types for the uninformed player. The situation is very different if there is a commitment problem as in Fearon, 'Fighting Rather than Bargaining'. Fearon shows that the use of offers by a Government to separate the weak insurgents from the strong is precluded (under certain conditions about frequency of offers) by the fact that the weak expect the Government to renege. Because of this commitment problem, the Government can only attempt to screen and separate the types by observing the result of fighting. If the weak hold out long enough, the cost to the Government of continuing to fight outweighs the benefits of making an offer that both weak and strong will accept. The Government will then make a pooling offer.

problems, we can no longer model the process of bargaining and fighting as a search for separation of types achieved through the well orchestrated, alternating, rounds of offers and counteroffers that characterize discrete time war and bargaining models. Indeed, in a discrete time framework, the players must wait for their turn. But postponing a decision until the next period means incurring a cost that is commensurate with the fixed delay imposed by waiting. This puts a premium on early resolution so that waiting is only worthwhile if it serves to resolve uncertainties, improving the terms of a final settlement. Any other rationale for temporizing is implicitly ruled out by the discrete-time framework. As a result, fighting stops once the players are made aware of their true prospects by the outcome of the fight and each other's observed bargaining behavior. This is no longer the case if we allow players to bargain and fight in the time continuum, making offers and counteroffers at the time of their choosing. As delay no longer imposes a *fixed* cost, although delay remains costly, the protagonists can decide to continue fighting in order to apply just enough pressure to persuade a weak opponent to surrender.[8]

To investigate war and bargaining in this context, we develop a continuous time model that allows the parties to make and accept offers at any time but allows the dispute to be settled on the battlefield should bargaining fail. The dynamic progression of the war is described by the standard Lanchester model. Our rivals are uncertain about each other's priorities, and war can lead one party to a victory, but it can also lead to a stalemate. We therefore relax many of the assumptions that are made in the war and bargaining literature. But most importantly, we create a game in which states can *choose* to remain silent, or not, while they fight.

[8]Unlike the continuous time framework, discrete time does not allow the players to manipulate the costs imposed by delaying tactics when using pure strategies. Using mixed strategies would allow such manipulation but would raise technical problems as well as the familiar interpretation issues that are associated to probabilistic strategies.

Using the standard perfect Bayesian equilibrium in Markov strategies[9] we find that while the players' types are revealed by their willingness to continue the fight they do not attempt to *actively* screen each other by making significant and separating offers. Instead, whatever the expected outcome of war, the rivals fight without making any bargaining concession for most of the conflict, preferring to wait it out in the hope that the other side will give in to outstanding demands. Only when one side's victory becomes imminent does the victor receive offers that are barely better than what he can guarantee himself by fighting to the finish. Moreover, our results continue to hold as the magnitude of uncertainty is reduced, either parametrically, at the modeling stage, or as the players gain information after the parameters are set and the game proceeds. In sum, *convergence of beliefs does not result from a convergence of offers* in our model. We predict instead that an enduring *divergence of positions* persists well into the last stages of the conflict.

The rationality of this uncompromising bargaining behavior hinges on the following calculus: the player who considers offering a compromise will weigh the lower war costs associated to an expected early settlement, against the opportunity cost of having the other side secure a better deal. In our model, war costs are not arbitrarily constrained by discrete-time assumptions and can be freely manipulated by the players' timing strategies. In this context, the trade-

[9]The standard subgame perfect (SPE) and perfect Bayesian (PBE) equilibria are merely Markov perfect (MPE) equilibria of one kind or another. All these perfectness concepts share the requirement of sequential rationality: each player maximizes his expected payoff at his turn of play given a current state of the game that summarizes his information. What distinguishes one type of equilibrium from another is how states, and the transition from state to state, are defined. In a PBE the transition is defined by Bayesian updating of beliefs. In a grim trigger SPE the transition results from observing a violation of expected play. MPEs are particularly attractive in repeated games because a few relevant states can summarize the various prior histories of play. The term MPBE refers to Markov perfect equilibria in repeated games when Bayesian updating of beliefs is involved, a necessary feature if we are to deal with uncertainty in a meaningful manner.

offs for both parties *strictly* favor an uncompromising bargaining stance unless one side's imminent victory forces the other to make incremental concessions to avoid complete defeat.

The fact that warring rivals *choose* not to compromise in equilibrium, regardless of the expected outcome of war or the magnitude of uncertainty, is a key finding for several reasons: firstly, the result suggests that a failure to compromise is linked to the anarchy of the bargaining environment between states at war. Indeed, our result appears when the well orchestrated bargaining protocols typical of other war and bargaining models are relaxed. Secondly, our analysis points to the importance of attrition behavior as a potent driver of war duration. Although our model is *not* a war of attrition model, as our rivals can seek war ending compromises instead of waiting each other out, attrition behavior *emerges* in equilibrium despite the screening and signaling potential of bargaining. Evidence that attrition behavior is a significant predictor of war duration has been established empirically by the authors.[10] Finally, the predicted failure of warring rivals to negotiate is in line with the observed behavior of states at war.

Convergence in the War and Bargaining Literature

James Fearon[11] identifies three reasons why states might decide to fight instead of negotiating in peace: uncertainty about each other's strength, commitment problems or indivisibilities in what is at stake between the warring parties.

[10] Catherine C. Langlois and Jean-Pierre P. Langlois, 'Does Attrition Behavior Help Explain the Duration of Interstate Wars? A Game Theoretic and Empirical Analysis,' *International Studies Quarterly* 53 (2009) pp. 1051-1073. We briefly discuss our empirical approach in the discussion section.

[11] James D. Fearon, 'Rationalist Explanations of War,' *International Organization*, 49 (1995) pp. 379-301.

However, Darren Filson and Suzanne Werner[12] point out that 'while all three explanations provide insight into why wars begin and endure, private information and incentives to misrepresent that information arguably explain best why belligerents who were unable to reach a settlement initially are able to do so later.' Explaining how protagonists who have private information can eventually reach an agreement after a fight is at the core of the Principle of Convergence. It requires that rivals bargain while they fight, and builds on the relationship between battle outcomes and bargaining positions as emphasized by Harrison Wagner.[13]

Convergence takes screening by the uninformed rival, and while the process is in some sense generic, the particulars vary quite widely according to model assumptions. Darren Filson and Suzanne Werner[14] provide a particularly crisp account of the process by which an attacker will calibrate his demands according to his beliefs about the defender's strength and the outcome of battle. In their model, fighting uses finite resources and this modeling choice distinguishes their model from Branislav Slantchev's.[15] In contrast to Filson and Werner, Slantchev allows for a process of offers and counteroffers so that the informed party has an opportunity to strategically take advantage of his rival's uncertainty in the setting of his own demands. While these authors emphasize the impact of ne-

[12]Darren Filson and Suzanne Werner, 'The Dynamics of Bargaining and War,' *International Organization*, 33 (2007) pp. 31-50.

[13]Harrison R. Wagner, 'Bargaining and War,' *American Journal of Political Science*, 44 (2000) pp. 469-484, and Harrison R. Wagner, *War and the State*, (The University of Michigan Press, Ann Arbor, 2007). Wagner discusses how the outcome of limited war can inform so that reasonable bargaining positions can emerge without a need to engage in fights to the finish. However, he does not derive equilibria of his war and bargaining model when uncertainty is present.

[14]Darren Filson and Suzanne Werner, 'A Bargaining Model of War and Peace: Anticipating the Onset, Duration and Outcome of War,' *American Journal of Political Science*, 46 (2002) pp.819-837.

[15]Slantchev, 'The Principle of Convergence in Wartime Negotiations,' *American Political Science Review*, 97 (2003) pp. 621-632.

gotiation outcomes as well as battle outcomes in the process of convergence, they develop their models assuming a small number of types. Slantchev distinguishes between three types and Filson and Werner assume only two.[16] The finite number of types assumption may not be entirely innocuous.

One of the distinct claims of the bargaining model of war is an integrated treatment of the 'onset, duration and outcome' of war.[17] Duration is determined by the time it will take to resolve uncertainty through screening. But if there is only a finite number of types, and screening is separating, maximum duration is predetermined by the model assumptions. The war can only last at most two periods in Filson and Werner[18] and at most three in Slantchev.[19] Of course, it would be simple enough, conceptually if not technically, to assume five types, twelve types, or any finite number N. But the issue of duration remains whole. Its upper bound is determined by the assumed characteristics of the process that is designed to reveal it, namely the sequenced separation of types, from the weakest to the strongest. There is an apparently simple solution to this problem. If we do not want a finite number of types, we can posit, as Robert Powell[20] does, that there is a continuum of types to screen. Then, it would seem, war duration is no longer constrained by the modeling of types.

In Powell's war and bargaining model,[21] the uninformed party is first described as screening the continuum of types according to a schedule that corresponds to the frequency of decision periods. Thus, if types are distributed evenly over the interval $[0, 1]$ it can be split into n pieces and the screening

[16] In 'The Dynamics of Bargaining and War,' Filson and Werner subsequently relax the two type assumption, multiplying the number of types and number of possible battles. They solve their model numerically.

[17] Filson and Werner, 'A Bargaining Model of War and Peace': Dan Reiter, 'Exploring the Bargaining Model of War,' *Perspectives in Politics*, 1 (2003) pp.27-43.

[18] A fact that is only reinforced by the assumption that resource constraints would only allow for at most two battles (Filson and Werner, 'A Bargaining Model of War and Peace').

[19] Slantchev, 'The Principle of Convergence in Wartime Negotiations.'

[20] Powell, 'Bargaining and Learning while Fighting.'

[21] Powell, 'Bargaining and Learning while Fighting.'

offers calibrated to the hardest to please within the subinterval. Conceptually, the limitations associated to the finite number of types assumption are reproduced in the choice of the frequency of decision making. This suggests further subdividing the period between battles, and Powell examines the impact of allowing for multiple offers within the interval of time that separates battles. The exercise illustrates the critical impact of the 'bargaining environment through which actors convey information.'[22] Indeed, if uncertainty is about costs (interpreted as costs of preparation for battle), multiplying the offers within a given time period allows for rapid screening and leads to immediate acceptance of a serious offer by the informed state.[23] By contrast, and this is Powell's point, if there is uncertainty about the chances of winning a battle, then fighting actually informs and screening may require at least one battle.

In sum, the finite number of types assumption imposes an artificial upper bound on war duration. But Powell's analysis suggests that if one assumes a continuum of types, the frequency of decision making can be accelerated enough to rule out duration altogether. This is the case when bargaining rather than fighting is the screening instrument. It would seem, then, that the Principle of Convergence is not entirely reliable when it comes to explaining how long wars last. To some extent, Alastair Smith and Allan Stam[24] escape these shortcomings by modeling the players as having heterogeneous priors that converge as battle outcomes are observed. Wars must necessarily last some time in Smith and Stam's framework as several forts must be won or lost before Bayesian updating can move the rival's divergent beliefs together. Convergence is achieved in Smith and Stam by observing the outcome of successive battles for forts but there is no opportunity in their model for strategic screening through bargain-

[22] Powell, 'Bargaining and Learning while Fighting,' p.251.

[23] This follows a result developed in Faruk Gul and Hugo Sonnenchein, 'Bargaining with One-Sided Uncertainty,' *Econometrica*, 56 (1988) 601-611.

[24] Alastair Smith and Allan Stam, 'Bargaining and the Nature of War,' *Journal of Conflict Resolution*, 48 (2004) pp.783-813.

ing. Moreover, the convergence of heterogeneous beliefs could be interrupted too early if war were to lead to a stalemate.

If rival positions become entrenched, fighting surely ceases to inform on relative strengths. How, then, can the convergence of beliefs be achieved? In fact, none of the war and bargaining models, except our own, allows for stalemate. This is because war is modeled as a series of battles that can be won with probability at each stage and lead inevitably to an outright victory by one rival or the other.[25] The workings of convergence if a stalemate is reached has been left un-addressed in the formal war and bargaining literature. Yet the United States remained trapped in a hurting stalemate in Vietnam and in Korea, while Iran and Iraq fought for eight years without making progress on the status of the Shatt-Al-Arab.

But, let us return to the fundamental modeling choices to be made when examining the workings of the Principle of Convergence. Powell[26] points out that screening *can* be accelerated by allowing quasi continuous decision making. However this is not a statement about what would happen if bargaining were allowed to take place in the time-continuum.[27] If instead of assuming that the players move in alternating time periods, however small, we assume that players can choose to move at any date in the time-continuum, would players *choose* to accelerate the screening process by multiplying offers within a short period of time? To investigate this and the related issue of convergence of beliefs as it affects war duration, our two-player model of bargaining and war in continuous time allows the rivals to make offers and counteroffers *at times of their choosing*,

[25] The Markov process has victory by one rival or the other as the only absorbing states.

[26] Powell, 'Bargaining and Learning while Fighting.'

[27] Letting the period separating successive turns approach zero is not equivalent to moving to the continuum. The logic of alternating turns together with the cost of waiting for one's next turn remain in the former but not in the latter. In continuous time, the timing of offers and acceptance becomes a decision variable and this erases the incentive to accept a given offer earlier rather than after the fixed and costly delay that is inherent to the alternating-turns framework.

while they fight.

Models of the bargaining process in continuous time can be found in the economics literature. Two prominent examples are Perry and Reny and Sakoviks.[28] While these authors allow agents to make offers at the time of their choosing, they constrain the bargaining process by imposing minimum delays between successive offers or between offer and acceptance. The equilibrium solution is then parametrized by these delays and this brings up the issue of how the players anticipate the other side's minimum delay. Moreover, the bargainers in Perry and Reny or Sakoviks are not at war. In war, delay means costly fighting but allows for more time to win on the battlefield. This creates incentives to settle or to fight that are not accounted for in the economics literature. Ponsati and Jarque *et al* [29] also propose using continuous time. But their bargaining framework is restricted to offers that are independently generated or that become public only when they match, assumptions that are not representative of the majority of wars. All these models predict that the protagonists will make serious screening offers to each other in equilibrium.

In our model, which imposes no restrictions on the timing or framework of bargaining, the players do not make serious offers to each other until one side's victory is imminent. Instead, for most of the war, each player waits for the other to be first to surrender. This we refer to as attrition behavior. In sum, we show that attrition behavior trumps bargaining in war until one side's victory is imminent. And attrition trumps bargaining even if stalemate is the outcome

[28] Motty Perry and Philip J. Reny, 'A Non-Cooperative Bargaining Model with Strategically Timed Offers', *Journal of Economic Theory*, 59 (1993) pp. 50-77: Jozsef Sakovics, 'Delay in Bargaining Games with Complete Information,' Journal of Economic Theory, 59 (1993) pp.78-95.

[29] Clara Ponsati, 'Compromise versus Capitulation in Bargaining with Incomplete Information,' *Annales d'Economie et de Statistique*, 48 (1997) pp. 191-210: Xavier Jarque, Clara Ponsati and Jozsef Sakovics, 'Mediation: Incomplete Information Bargaining with filtered Communication,' *Journal of Mathematical Economics*,39 (2003) pp.803-830.

on the battlefield.

Conceptual Issues and Model Assumptions

The emergence of attrition behavior and the failure of bargaining when states at war can *choose the timing* of offers and counter-offers is a provocative result that runs counter to the established wisdom embodied in the Principle of Convergence. To what extent do our modeling assumptions drive our conclusions? Is our result an artifact of continuous time modeling? And is it reasonable to assume that state actors make decisions in continuous time?[30]

Are states truly free to make or accept offers at the time of their choosing? From a practical standpoint, one could argue that discrete time modeling is appropriate because decision makers need some time to respond to offers or formulate counter-offers and need, throughout the war, to pause to reevaluate their plans in light of observed developments. Such reasoning is technically and conceptually flawed. Technically, optimal acceptance times in our model are given by continuous probability distributions. This means that an instantaneous acceptance has probability zero and can therefore never happen. And the probability that the players receive a very fast response to their offers is commensurately very small. Decision makers will therefore take some time to respond to an offer with high probability. Moreover in a game theoretic framework, whether in discrete or continuous time, strategies are complete contingency plans made by the players at the very start of the game. Players do not pause to *reevaluate* their plans in equilibrium. We therefore rule out this line of argument as a rationale for adopting a discrete time framework.

From a technical standpoint, it is important to note that continuous time modeling does not mean that players make decisions at each instant. Continuous

[30] This discussion is inspired by the extensive and insightful feedback received from one of the anonymous reviewers of this article.

time decision making is not to be interpreted as what happens when the length of the decision period in a *discrete time* model is made arbitrarily small. Instead, the continuous time framework avoids the theorist-imposed fixed cost of delay incurred if players wait till the next period to take action. It merely allows the protagonists to *choose* the timing of their bargaining moves and the pace at which they screen the adversary for resolve by making serious settlement offers. The real issue is whether it is in their best interest to do so. Our core finding is that it is not. Rational players prefer to make zero offers for most of the war letting the other side's willingness to surrender be the only source of information on type.

Are our findings on bargaining failure and attrition behavior artifacts of continuous time modeling? To answer such a question, we would need to be sure that our results cannot emerge from a discrete time model. The competing discrete time models search for *separating* equilibria in contexts that are arguably quite restrictive. The most sophisticated discrete time models by Slantchev and Powell[31] allow only one-sided uncertainty. And Powell allows only one side to make offers while Slantchev allows only three types. These assumptions drive their convergence results. In order to assess the relative merits of the discrete and continuous time frameworks, one would need to considerably extend the former, allowing two-sided uncertainty, bilateral bargaining, and an unrestricted number of types, as we do. And importantly, one would need to consider *semi-separating* equilibria in order to allow the players to manipulate costs at will. But even under the restrictive assumptions adopted in existing discrete time models, the technical challenges involved in the search for semi-separating equilibria are daunting.[32] In fact, the technical flexibility of continuous time

[31]Slantchev, 'The Principle of Convergence in Wartime Negotiations'; Powell, 'Bargaining and Learning while Fighting.'

[32]In a fully generalized discrete time model, it is not clear that separating equilibria can be obtained let alone semi-separating equilibria that would be even harder to obtain given the non-linearity of Bayesian updating. In the continuous-time framework, by contrast, we

modeling has been one of the primary motivations in the choice of our modeling framework. Without compelling evidence to the contrary, we have no reason to believe that it *creates* the results that we observe.

We now turn to our model and its consequences.

The Model: Timing, War Dynamics, Costs and Bargaining

Our main objective is to understand how the convergence of beliefs about the rivals' war costs or priorities affects the bargaining process when the protagonists can make potentially war ending settlement offers at any time. To this end, we build a model that combines three elements: a continuous-time framework that allows all moves by each side and all developments of the war to occur at any point in time; a dynamic depiction of the war's progress using the standard Lanchester equations; and a bargaining protocol that allows each player to define alternative outcomes by making and updating offers at any point in the game. The definition of player objectives as well as the solution concept used reflect all these elements.

War Dynamics

War can be conceptualized as a continuous process of loss and reinforcement that can lead to a variety of outcomes ranging from outright victory to stalemate on the battlefield. Its formal implementation takes the form of a system of differential equations as pioneered by Lanchester.[33] The standard Lanchester

can use the calculus of integrals which is a far more powerful mathematical tool than the algebra of power series that must be used in discrete time models. This is, in fact, the main motivation for our modeling choice.

[33] Frederick W. Lanchester, 'Aircraft in Warfare: The Dawn of the Fourth Arm-No V. The Principle of Concentration,' *Engineering*, 98 (1941) pp.422-423.

model is deterministic, so that the players know in advance whether they are headed for the victory of one side or the other, or for a military stalemate. Uncertainty in our model is about resolve rather than capabilities as the deterministic war model informs the players about their ultimate prospects in war.[34] Assuming a deterministic path for war should favor use of the bargaining option, especially if one side is expecting defeat. Yet, in equilibrium, we will find that the protagonists prefer to remain uncompromising for much of the war.

The basic variable involved in the process of war is a force level $z_i(t) \geq 0$ for side i at time t. Player i's forces have a maximum size $b_i > 0$ that is assumed constant for the duration of the war and the difference $(b_i - z_i)$ can be mobilized at rate $a_i \geq 0$. In the absence of combat losses, z_i would grow at rate $z_i' = a_i(b_i - z_i)$ from an initial size $z_i(0) = z_i^0$. But once combat has started, the opponent j's forces can destroy, disable or capture i's forces at a rate $k_i z_j$ that is proportional to its own forces. The factor k_i can be interpreted as i's vulnerability to j's attacks or as the capability of j. The resulting evolution equations read, for the two sides, as long as z_i and $z_j \geq 0$,

$$z_i' = a_i(b_i - z_i) - k_i z_j \tag{1}$$

and symmetrically for z_j'. This system of differential equations is formally identical to the so-called generalized Lanchester equations that can be found in Bellany.[35]

The evolution of the system depends on the quantities $q_i = (a_i b_i - k_i b_j)$ that measure the difference between the maximum growth rate $a_i b_i$ of i's forces and destruction rate $k_i b_j$ by j's forces. There are three generic cases:

(a) if $q_i > 0$ and $q_j > 0$ then the system evolves towards a stalemate (z_i^*, z_j^*), with $z_i^* = \frac{a_j(a_i b_i - k_i b_j)}{a_i a_j - k_i k_j}$ from any starting values;[36]

(b) if $q_i > 0$ and $q_j < 0$ (conversely $q_i < 0$ and $q_j > 0$) then the sys-

[34] Uncertainty on capabilities would require adding stochastic terms to the Lanchester war model, an option that goes beyond the scope of this paper.

[35] Ian Bellany, 'Modeling War,' *Journal of Peace Research*, 36 (1999) pp.729-739.

[36] This also assumes $a_i a_j - k_i k_j > 0$.

tem evolves toward a complete annihilation of j's forces (conversely i's forces) starting anywhere;

(c) if $q_i < 0$ and $q_j < 0$ then the system evolves generically to the complete annihilation of one side or the other depending on the current force ratio.

Using the generalized Lanchester equations to describe the players' fate on the battlefield allows us to consider the three most interesting battlefield outcomes: In case (a) war will not resolve the dispute because the fighting will lead to a stalemate; in (b) war will inevitably lead to the victory of one well identified side; and in (c) initial force levels play decisive roles in determining the outcome.

Costs and Uncertainty

We assume that each side is uncertain about the other's willingness to continue the fight, despite the costs of battle and the perspective of defeat or hurting stalemate on the battlefield. We therefore introduce bilateral uncertainty about each other's resolve. We assume that costs reflect the maintenance and supply of a fighting force, the casualties inflicted by the enemy as well as possible collateral damage to property and society. Although our theory is not limited to this case, a typical formulation of such costs can be written:

$$c_i(t) = \alpha_i z_i(t) + \beta_i z_j(t) + \gamma_i \tag{2}$$

with α_i, β_i and $\gamma_i \geq 0$. The term $\beta_i z_j(t)$, which is commensurate with the other side's force level, reflects war casualties and at least some of the collateral damage. We also assume that costs fall to zero when hostilities end.

To model uncertainty about costs we assume that $c_i(\tau) = (1 - \epsilon i)c_{\mathcal{I}}(\tau)$ for type $i \in \mathcal{I} = [0, 1]$ with $c_{\mathcal{I}}(\tau) \geq 0$ common to side $\mathcal{I}$ and where $\epsilon \in (0, 1)$ is the initial magnitude of uncertainty. As an alternative, we could assume that some parameters of costs are common to all types while others are not. For instance β_i might read $\beta_i = (1 - \epsilon i)\beta_{\mathcal{I}} > 0$ while all other elements of the cost function are common to all types. This would focus uncertainty on the cost of casualties. In all cases, a higher value of i means a type that is less sensitive to the costs of war and therefore more resilient. In general, we assume that the very definition of types ensures that they are ordered by increasing strength so that $\frac{\partial c_i}{\partial i} < 0$ for all times t. For simplicity of presentation, we also assume a uniform initial distribution of the types.

Offers and the Bargaining Protocol

We assume that if one side's forces, $\mathcal{I}$ for instance, are annihilated at some time θ, meaning $z_i(\theta) = 0$,[37] then the victorious side $\mathcal{J}$ receives the entire prize in dispute and the game ends in that perpetual outcome. But we also allow each side to offer the other any share of that prize while they fight.[38] For instance, side $\mathcal{I}$ can offer side $\mathcal{J}$ any share $y_i(t) \in [0, 1]$ at any time t and can change it at any subsequent date as long as it has not been accepted. An offer $y_i(t) = 0$ is valid and is interpreted as 'no offer' or as a demand for surrender. An offer can be accepted only after it is made and as long as it remains available. We impose no constraint on how soon the acceptance or withdrawal of an offer can be made.[39] But, since an offer made and withdrawn at the very same point in time could never be accepted we require that it remains open for some time, however brief, after it is made. This means that $y_i(t)$ must remain constant or evolve smoothly within some interval $[t, t')$ where $t' > t$ is chosen by side $\mathcal{I}$.[40]

Surrender, or more generally the acceptance by type $j \in \mathcal{J}$ of an offer $y_i(t)$ at time t, whether it is a zero offer or not, ends the game. It yields the

[37] z_i is common to all types i on side $\mathcal{I}$.

[38] We, of course, assume that the prize is divisible at will or that compensations, monetary or other, allow such divisions.

[39] By contrast, Perry and Reny, 'A Non-Cooperative Bargaining Model' or Sakovics 'Delays in Bargaining Games' impose reaction delays. Our choice is motivated by our criticism that set unit periods exclude by design explanations other than uncertainty for war duration.

[40] Technically, $\mathcal{I}$'s terms will be fully described by a function y_i of time t that can be left-discontinuous but is right-continuous with a piecewise continuous derivative.

constant utilities $u_i(1-y_i(t))$ and $u_j(y_i(t))$ to types i and j respectively for all future time $s \geq t$.[41] Utilities are strictly increasing in one's share. Technically, we assume that $u_i(0) = 0$, and $u_i'(x) \geq u > 0$ for all i and x. If we choose to focus uncertainty on utilities rather than on costs it can be modeled by $u_i(x) = (1+\epsilon i)u_{\mathcal{I}}(x)$ with $u_{\mathcal{I}}(x)$ common to side $\mathcal{I}$ and where $\epsilon \in (0,1)$ is the initial magnitude of uncertainty.

Expected Utilities

The players in our model consider their discounted flow of utility to infinity as a guide to the decisions that they want to make. These farsighted players will consider the possibility that the war will end with a negotiated settlement, or the surrender of the enemy. And they will take the deterministic path of war and its costs into account when deciding whether to make a settlement offer at all. The expected utility that each player seeks to optimize must be defined to account for all possible contingencies and this is what we pursue now. Consider the possibility that the game involves fighting with cost flow $c_i(\tau)$ from an initial time arbitrarily set to $t = 0$ until an outcome yielding share x_i to side $\mathcal{I}$ is reached at time s. Share x_i can take any negotiated value in $[0,1]$ and is set to $x_i = 1$ in case of military victory by i and to $x_i = 0$ in case of i's military defeat. Type i's discounted payoff under these circumstances can be written:

$$\pi_i(x_i, s) = -\int_0^s r_i e^{-r_i \tau} c_i(\tau) d\tau + e^{-r_i s} u_i(x_i) \qquad (3)$$

where $r_i > 0$ is a discount parameter that characterizes type i or is common to side $\mathcal{I}$. The integral in (3) represents the cumulated costs of fighting until date s, appropriately discounted, while the second term measures the discounted

[41] We also assume that an overgenerous offer $y_i(t) \geq 1 - y_j(t)$ by i at time t means an immediate acceptance of $y_j(t)$ and that simultaneous acceptances of mismatched offers have no effect. Making no explicit offer is interpreted as offering $y_i = 0$. If the game ends at time θ in the military victory of one side the offers of the two sides are trivially set to the corresponding outcome (e.g., $y_i \equiv 0$ for $t \geq \theta$ if i wins) for notational consistency.

utility gained from enjoying share x_i in perpetuity.

Let $\phi_j(s)$ denote the probability that some type $j \in \mathcal{J}$ concedes, meaning surrenders or accepts side $\mathcal{I}$'s offer $y_i(s)$, by time s. And let $d\phi_j(s)$ be the probability of that development within an infinitesimal interval ds at time s.[42] Waiting time t for type i to concede then yields the expected payoff:

$$\mathbb{E}\pi_i(t) = \int_0^t \pi_i(1-y_i(s), s) d\phi_j(s) + (1-\phi_j(t))\pi_i(y_j(t), t) \tag{4}$$

The integral term is the value associated to an expected concession by j before time t and the second term is the value to i of accepting j's standing offer at time t.[43]

The above probability $\phi_j(t)$ corresponds to the belief, common to all types $i \in \mathcal{I}$, about which types $j \in \mathcal{J}$ should have conceded by time t.[44] Indeed, types $j \in \mathcal{J}$, ordered by increasing resilience, are arrayed uniformly on the interval $[0, 1]$. As time passes, and given side $\mathcal{I}$'s standing offers, $y_i(s)$ over interval $[0, t)$, failure of side $\mathcal{J}$ to accept them is indicative of side $\mathcal{J}$'s true type. Thus if $[0, \phi_j(t)) \subset \mathcal{J}$ denotes the interval of types j that all i believe have been screened out before time t, the non-decreasing $\phi_j : \mathbb{R}_+ \to [0, 1]$ is a probability distribution function. The remaining types $j \in [\phi_j(t), 1]$ are active at time t and still remain to be screened.

The Best Alternative to Negotiation

To be acceptable to an active type i, an offer $y_j(t)$ by side $\mathcal{J}$ must be at least as good as what i can guarantee himself by foregoing negotiations and fighting to the finish. If i is facing an inevitable defeat or a stalemate, he can guaran-

[42]The notation $d\phi_j(s)$ rather than $\phi_j'(s)ds$ reflects the fact that ϕ_j can be discontinuous. At discontinuities $d\phi_j(s)$ is a mass with magnitude equal to the size of the jump.

[43]Implicit in (4) is the convention that $\pi_i(x_i, s) \equiv \pi_i(x_i, \theta)$ for $s \geq \theta$ if a military outcome x_i is reached at time θ. Therefore $\mathbb{E}\pi_i(t) \equiv \mathbb{E}\pi_i(\theta)$ for all $t \geq \theta$ since, also by convention, $x_i \equiv 1 - y_i(s) \equiv y_j(t)$ for all $t \geq s \geq \theta$.

[44]In the case of a non-uniform initial distribution of types j, the probability ϕ_j would involve the beliefs about side $\mathcal{J}$ as well as the distribution of types.

tee himself nothing of value and can only eliminate costs by surrendering and accepting a share $x_i = 0$. This is his best alternative to negotiation. Player i could instead be headed for imminent victory so that the utility of winning the entire prize, net of the remaining costs incurred to fight for it, is positive. His best alternative to any settlement would then be given by the expected value of fighting to the finish without talking. But this only occurs if victory is imminent so that the costs of fighting to the finish no longer overwhelm the discounted value of holding the prize in perpetuity. For most of the war then, the best alternative to negotiation for the winning side is also to surrender.

Formally, player i's best alternative to negotiation is the maximum share of the prize that is no better for him than fighting to the finish. It is not i's expected payoff as the expected payoff accounts for the possibility that one or the other side will accept a standing offer. If i expects the battle to reach a victory for one side at time θ, and to involve cost flow c_i, then that share $\xi_i(t)$ at time $t \leq \theta$, if positive, is the payoff equivalent of fighting for time $(\theta - t)$ to reach the outcome x_i ($x_i = 1$ if he wins, $x_i = 0$ if he loses). Formally

$$u_i(\xi_i(t)) = \max\{0, -\int_t^\theta r_i e^{-r_i(\tau - t)} c_i(\tau) d\tau + e^{-r_i(\theta - t)} u_i(x_i)\} \tag{5}$$

If i expects to win, $\xi_i(t) = 0$ for most of the war but turns positive as victory nears. Since a stronger i always has a lower cost flow, $\xi_i(t)$ will turn positive sooner for the stronger winning side. But the loser's best alternative to negotiation remains zero regardless of type. If $\theta = \infty$ because of an expected stalemate, then the best alternative to negotiation is nil for both sides regardless of type. In sum, $\xi_i(t) = 0$ for both sides, regardless of type, for most of the war. It is important to keep this in mind as we show that, in equilibrium, it is best for each of the players to offer *no more* than the other's best alternative to negotiation until victory is near for one side or the other. This means that for most of the war it is optimal for each side to offer *nothing* and to rely on time passing, and the resulting observation of the other side's decision to surrender or not, for screening.

A Numerical Example

To illustrate the link between the dynamics on the battlefield and the bargaining range available to the players we work out a numerical example that will illustrate how the best alternative to negotiation varies with type.

First let $a_i = a_j = 1$, $b_i = b_j = 1$ and $k_i = k_j = 2$ in (1). Further assume that initial force levels for i and j are $z_i^0 = \frac{3}{5}$ and $z_j^0 = \frac{1}{2}$ (both less than the maximum of 1). With these parameter values, each side's forces evolve according to (1) as follows:

$z_i(t) = \frac{1}{6}(2 + \frac{13}{10}e^{-3t} + \frac{3}{10}e^t)$ and $z_j(t) = \frac{1}{6}(2 + \frac{13}{10}e^{-3t} - \frac{3}{10}e^t)$

$z_j(t)$ continually decreases until j's defeat at time $\theta \simeq 1.90$ where $z_j(\theta) = 0$. $z_i(t)$ initially decreases until time $\tau \simeq 0.64$, then increases until time θ when it reaches $z_i(\theta) \simeq 0.67$ at which time the war ends in i's victory unless one or the other party has given in to the other's demands rather than continue the fight.

As the war proceeds and i's victory becomes imminent, the best alternative to negotiation $\xi_i(t)$ also evolves. We assume further that discount parameter $r_i = 1$ and that uncertainty is about costs with the range of costs for side $\mathcal{I}$ given by:

$c_i(z_i, z_j) = (2 - i)(z_i + z_j)$

Since side $\mathcal{I}$ is headed for victory at time θ the best alternative for any type i will become positive as t approaches θ. The stronger types i have lower costs and therefore experience a higher payoff from fighting to the finish. For instance, $\xi_i(t)$ becomes positive at $t \simeq 1.0$ for $i = 1$ while $i = 0$ must wait until $t \simeq 1.34$ for this to happen. In this example, even the strongest type on side $\mathcal{I}$ has a zero best alternative to negotiation for more than half of the war. The situation is pictured in Figure 1 below.

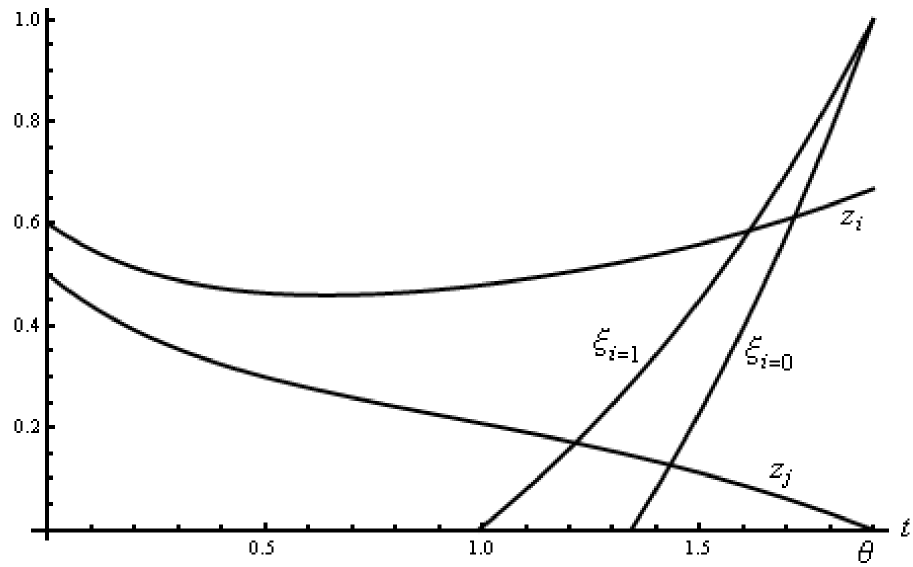

Figure 1: A Victory for Side $\mathcal{I}$

Schedules z_j and z_i illustrate the evolution of each of the players' forces over time. If the players fight to the finish, j sees its forces decrease consistently over time until they are entirely depleted at date θ. By contrast, i, the victor can expect an initial decrease in his forces to be followed by growth as he approaches complete victory on the battlefield. Meanwhile, as i approaches victory, fighting to the finish eventually yields a positive expected payoff as the discounted value of the prize exceeds remaining battle costs. In other words, side $\mathcal{I}$'s best alternative to a negotiated agreement turns positive to match its expected payoff of a fight to the finish. Side $\mathcal{I}$'s best altenative is represented by schedules $\xi_{i=1}$ for the strongest type on side $\mathcal{I}$ and by $\xi_{i=0}$ for the weakest. The weakest type on side $\mathcal{I}$ incurs the highest war costs. The expected value of fighting to the finish therefore turns positive later for type $i = 0$ than it does for the strongest type $i = 1$. As illustrated in Figure 1, type $i = 1$'s best alternative, schedule $\xi_{i=1}$, turns positive at $t \simeq 1$ while for type $i = 0$ this does not happen until $t \simeq 1.34$. Past these dates both $\xi_{i=1}$ and $\xi_{i=0}$ grow as remaining battle costs decrease and the discounted value of the prize grows as it becomes increasingly within close reach. As the war ends with j's defeat in battle, schedules $\xi_{i=1}$ and $\xi_{i=0}$ converge to the present value of the entire prize, set to 1.

As j is headed for defeat, $\xi_j(t) \equiv 0$ at all dates t whatever his type. The range of offers that could be rationally entertained by the two sides should therefore begin to shrink at some time $t \in (1, 1.34)$, depending on how much uncertainty has been resolved, until it reduces to the point $\xi_i(\theta) = 1$ and $\xi_j(\theta) = 0$ at victory time $\theta \simeq 1.90$. How will war end and what concessions will the rivals make? This is what we find out by examining the equilibria of our bargaining and war game.

War and Bargaining in Equilibrium

Following much of the formal literature we use the perfect Bayesian equilibrium in Markov strategies (MPBE) as solution concept. To investigate the convergence of offers and beliefs and its role in determining the duration of wars, we focus on separating equilibria, that allow the players to learn about each other's types by observing behavior.[45]

The Findings

The formal analysis of our model is presented in appendix and proceeds in three phases. We first obtain the optimal acceptance time for each type, given the two sides' offers, and the corresponding consistent beliefs. We then characterize optimal offers given consistent beliefs. We finally show the existence of a unique

[45]While pooling equilibria may exist, players would not learn about each others' types, convergence could not take place and the issue of convergence would be moot. And semi-separating equilibria would picture each individual type making probabilistic decisions, a feature that is unnecessary given our continuum of types and would raise interpretation problems.

MPBE. The properties of that equilibrium are found to be incompatible with the Principle of Convergence. Offers fall into three classes:

First, the offers that we call convergent drive a *separation* of types and are *significant* as they grant the other side a positive share that typically exceeds their best alternative to negotiation. For example an offer schedule that would propose a gradually increasing share of the prize over time, with the increase calibrated to encourage types on the other side to accept an early settlement rather than fight to the bitter end would be a convergent offer. It would accelerate the screening of types and reduce the duration of the war. Resolution of uncertainty through bargaining would then be determining of how long the war lasts. An equilibrium involving bilateral convergent offers would therefore support the Principle of Convergence. Indeed, convergence requires that offers actually be made and that the resulting screening process ensures that the players learn from their opponent's response to their offers.

Second, at the boundary of the set of convergent offers lies the set we simply refer to as 'boundary offers.' Such offers may be either separating or significant, but they cannot have both features. If a boundary offer drives a separation of types it must match the other side's best alternative to a negotiated settlement. As long as fighting to the finish is more costly than the discounted value of the prize enjoyed in perpetuity, the best alternative to a negotiated settlement is nil, the boundary offer is also nil and types are revealed by the simple passage of time for those unwilling to continue the fight. If, instead, a boundary offer is significant it is because the other side is nearing victory and enjoys a positive best alternative to negotiation. This significant offer then has the effect of hiding the types of the losing side that makes it and therefore fails to be separating.

Third and last, we define the set of offers that are unacceptable. For example, an offer that falls below the other side's best alternative to negotiation would never be accepted by a rational player and therefore joins the set of unacceptable offers. But there are other types of offers that that would never be rationally accepted. It turns out that offers that increases too fast or that involve discontinuities are also unacceptable because they prompt the other side to wait with certainty for a while.

On the way to building the proof of our main theorem, we obtain two substantive results. The first result (Lemma 8 in appendix) asserts that a convergent offer can always be profitably reduced by the side that makes it. The intuition behind this result is as follows: a substantial offer accelerates the screening of the other side's type by prompting each type on the other side to end the war sooner than it would by relying on its resilience alone. Thus accepting the substantial offer reduces war costs by shortening the fight. But it turns out that the reduced war costs for the side making the compromise offer do not compensate for the receiving of a reduced share of the prize and this speaks in favor of reducing the size of the offer. Indeed, the compromising side could, with some probability, be facing a type on the other side who would have surrendered right away. And the expected gain from reduced war costs does not match the expected loss from giving away part of the prize to a possibly weak type on the other side. The second result (Lemma 9) implies that optimal offers can never match, meaning that there is always a spread between the two sides' proposals, unless and until one side reaches victory. As a result, in this continuous time framework, it is not rational to reach an early compromise without fighting. These two first results lead to our central theorem that establishes the existence of an equilibrium that must involves *boundary* offers on each side.

In sum, equilibrium behavior in our model involves standing firm and making no concession for most of the war. In fact, if there is stalemate, optimal behavior dictates that neither side *ever* makes any concession despite the ongoing convergence of beliefs. As a result, the game turns into a pure war of attrition despite the possibility of negotiation. The Principle of Convergence is clearly not involved in determining such wars' duration. Instead, these wars will last until one side surrenders to the other giving up the entire prize. If one side is expected to win, no concessions are made until the winner's expected utility from a fight to the finish is positive. Only then will the losing side make offers that affect the duration of the war by prompting the winning side to accept a settlement rather than fight to the bitter end. Thus for much of these wars neither side concedes anything to the other and duration is determined by the resilience of the types on each side not by screening as the Principle of Convergence suggests. For this initial phase of the war, the Principle of Convergence is *not* involved in determining duration. When victory becomes imminent, then screening by the losing side does affect war duration but it does so through a process that is arguably very different from the one that is typically envisaged by the convergence principle. In a discrete time model with bilateral uncertainty, the workings of Principle of Convergence would dictate a process of alternating offers and counteroffers designed to satisfy the weakest type still standing on each side. And the process would begin with the start of hostilities. In our model, when victory becomes imminent, the losing side actually hides its type by ruling out surrender in favor of extending offers that could prompt the weaker types on the winning side to settle rather than fight. The intent for the losing side is to try to save something for himself now that the winner has revealed herself to be resilient enough to prefer a fight to the finish. The objective is not to reveal the winner's type but to salvage what might be possible from the ruins of defeat.

We now turn to a numerical example to illustrate these findings.

A Numerical Example

Consider again our example in which $\mathcal{I}$ is the winning side. Also assume that utilities are given by $u_i(x) = x$. Finally assume symmetric parameters for side $\mathcal{J}$, except that j's costs are given by $c_j(z_i, z_j) = (3-j)(z_i + z_j)$. Each type $i \in \mathcal{I}$ has an expected payoff $\mathbb{E}\pi_i(t)$ defined by (4) given the two sides' respective offers and beliefs. We illustrate expected payoffs for the three types $i = 0, 0.5$ and 1, given side $\mathcal{J}$'s optimal offer y_j which we will describe in Figure 3 below. We consider two possible offers $y_i \equiv 0$ and $y_i \equiv 0.20$ by side $\mathcal{I}$ and trace out, in Figure 2, the expected payoff to the three types of waiting time t to accept side $\mathcal{J}$'s outstanding offer. Expected payoffs on the veritical axis are measured against waiting times on the horizontal t axis.

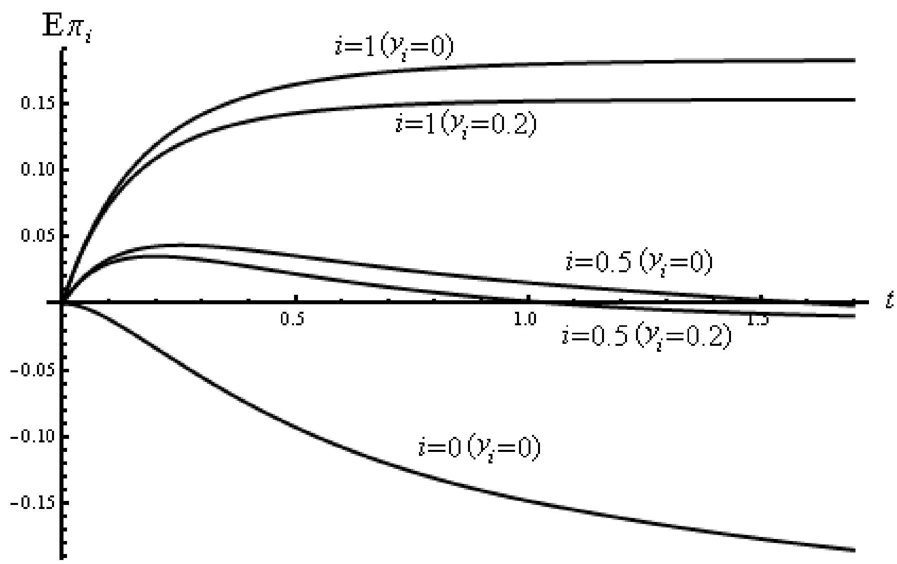

Figure 2: Expected Payoffs

Figure 2 illustrates two things: First, although $\mathcal{I}$ is the winning side, the weaker individuals on side $\mathcal{I}$ will prefer to give in to side $\mathcal{J}$'s offer before victory rather than fight to the finish. For example, type $i = 0$ experiences negative expected payoffs from the start and so would be better off accepting side $\mathcal{J}$'s standing offer at time $t = 0$, which is $y_j(0) = 0$. This is of course an extreme case and expected payoff schedules shift continuously as i becomes more resilient. Type $i = 0.5$ would give in to the extreme demand $y_j = 0$ by side $\mathcal{J}$ around $t = 0.25$, assuming i made no concession, and weaker types $i \in [0, 0.5]$ would give in to $y_j = 0$ sooner. Indeed, Figure 2 shows that, given $y_i \equiv 0$, type

$i = 0.5$'s expected payoff from waiting time t to accept $\mathcal{J}$'s standing offer reaches a maximum around $t = 0.25$. Type $i = 1$, the most resilient, actually finds it optimal to fight to the finish when offering $y_i \equiv 0$, and only until offers match when offering $y_i \equiv 0.20$.

Secondly, given $\mathcal{J}$'s optimal offer y_j, no type on side $\mathcal{I}$ has any interest in extending a positive offer y_i to $\mathcal{J}$. Observe that, given type, the expected payoff associated to the offer $y_i \equiv 0.20$ is always lower than the expected payoff associated to $y_i \equiv 0$. Given $\mathcal{J}$'s optimal offer, all types on side $\mathcal{I}$ strictly prefer to stand firm on initial demands as their expected payoffs shrink with any concession made to the other side. Figure 2 pictures the situation at the onset of the war. As time passes, types $i \in [0, \phi_i(t))$ are screened out and all expected payoffs for $i \in [\phi_i(t), 1)$ shift down and to the left, with $i = \phi_i(t)$ taking the place of $i = 0$ and seeing his expected payoff decrease from that time on. We now illustrate optimal behavior.

As seen in section 2.4, side $\mathcal{I}$ ultimately wins. As a result, if war drags on up to time $t \simeq 1$, active types see their best alternative to negotiation turn positive and side $\mathcal{J}$ starts optimally making concessions. But until then side $\mathcal{J}$ makes no concession whatsoever. With side $\mathcal{I}$'s victory approaching side $\mathcal{J}$ attempts to salvage some of the prize for himself by making an optimal offer that must satisfy $y_j(t) \geq \xi_i(t)|_{i=\phi_i(t)}$ which involves a concession to some of the active types $i \geq \phi_i(t)$ on side $\mathcal{I}$. While the winding down of the war comes with concessions from side $\mathcal{J}$, uncertainty about i's type narrows. This can be seen by observing ϕ_i as it changes over time. Figure 3 plots ϕ_i, ϕ_j, and equilibrium offer y_j together with the best alternatives to negotiations ξ_i for types $i = 0.878$ and $i = 1$. Equilibrium offer $y_i \equiv 0$ for side $\mathcal{I}$ is not shown but is implicit in the picture.

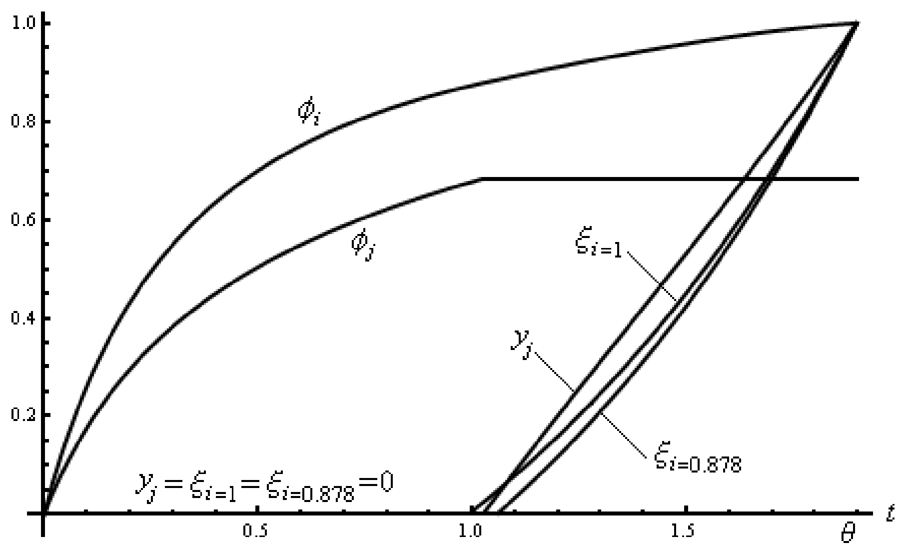

Figure 3: Beliefs and Optimal Offers

Optimal offer y_j starts out being identically nil, just as all ξ_i's are, but turns positive as victory for side $\mathcal{I}$ is looming. At time $t^* \simeq 1.03$, type $i = \phi_i(t^*) \simeq 0.878$ accepts what is still a nil offer $y_j(t^*) = 0$. But immediately after t^* optimal offer y_j becomes positive. y_j initially lies below the strongest type's $(i = 1)$ best alternative $\xi_{i=1}$ but somewhat above $\xi_{i=0.878}$. So, for $t > t^*$ there is a range of types $i \in (0.878, 1)$ for whom the optimal $y_j(t)$ is *strictly above* their best alternative $\xi_i(t)$ and is therefore a small concession. Offer y_j increases at a rate that results in the *constant* consistent belief ϕ_j as all remaining active types j adamantly refuse to accept side $\mathcal{I}$'s optimal offer $y_i \equiv 0$.[46] Offer y_j is therefore a boundary offer as it is either nil or prevents the separation of types on side $\mathcal{J}$.

The fact that side $\mathcal{J}$'s optimal offer increases over time leads the stronger types on side $\mathcal{I}$ to wait before accepting, although the current offer exceeds

[46]The positive part of y_j satisfies the boundary condition $\lambda_{i=\phi_i} \equiv 0$ discussed in appendix (Definition 2).

their current best alternative to negotiation. Recall that side $\mathcal{I}$'s acceptance time is determined by maximizing its expected payoff from waiting to accept $\mathcal{J}$'s optimal offer as illustrated in Figure 2. Expecting better offers to come their way, active types on side $\mathcal{I}$ will optimize their expected payoffs by fighting longer, delaying acceptance, and after time t^*, active types i will be accepting deals that are *strictly* better than fighting to the finish. Of course, the stronger types will only accept the more generous offers. For example, types i close to 1 will only accept an offer that is close to the entire prize. And the strongest type on side $\mathcal{I}$, $i = 1$, will fight to the finish for the entire prize.

Discussion

The uncompromising bargaining behavior that emerges from the our results persists as each side gathers more information about the type he is facing. Beliefs are converging in our model but it is only the imminent victory of one side that prompts significant offers by the losing side, and these offers are a little better than what the impending winner can guarantee himself by fighting to the finish. Our model therefore predicts that the winning side could well stop short of a complete military victory and accept the loser's attempt to limit the damage. The victor would then agree to a little less than the entire prize having already achieved most of his objectives. For example, the United States and its allies stopped short of marching into Baghdad in the First Persian Gulf War, preferring instead to withdraw the troops having achieved the liberation of Kuwait and the clear defeat of Iraqi troops in the Desert Shield ground offensive. By bringing the troops home leaving Saddam to extoll Iraqi courage in the 'Battle of Battles (Um Al-Ma'arik), where Iraq stood fast against the invasion, maintaining its sovereignty and political system'[47] the United States' was arguably accepting less than complete victory, having achieved most of its

[47]http://www.globalsecurity.org/military/ops/desert_storm.htm

objectives.

By contrast, if the war evolves to a stalemate the model predicts that one side will eventually cave in and accept the other side's position, although this may take a very long time. The Korean war is a case in point. As Carter Malkasian points out,[48] General Ridgeway commander of the Eighth Army, and Peng Dehuai, commander of the Chinese People's Volunteers, 'both envisaged a protracted war, seeking to compel concessions from the opponent.' Following what Andrew Birtle[49] describes as the three years of stalemate, North Korea conceded the right to voluntary repatriation of those who had escaped the communist regime, after conceding the 1951 line of contact between the warring armies as the cease fire line. According to Malkasian, within the context of limited U.S. war aims designed to avoid an escalation to 'a general war between the US and the Soviet Union or the People's Republic of China, 'the stalemate in Korea ended with North Korea's full back down.[50]

As emphasized by Allan Stam[51] we model war as an effort by *both* sides to coerce the other to adopt his point of view. What our model predicts, however, is that if war is a bargaining process it is one that is indeed what Thomas Schelling would describe as 'dirty, extortionate and often quite reluctant.'[52] The screening implied by our no-concession result stands in stark contrast with the screening process that underlies the Principle of Convergence. Convergence in Filson and Werner, Slantchev, or Powell[53] is achieved by calibrating offers *in each decision period* to the weakest type still standing. As information becomes

[48] Carter Malkasian, *A History of Modern Wars of Attrition*, (Praeger, Westport Connecticut, London, 2002).

[49] Andrew Birtle, *The Korean War: Years of Stalemate July 1951 July 1953*, (Korean War Commemorative Brochure, U.S. Army Center of Military History, last updated 2007)

[50] Malkasian, *A History of Modern Wars of Attrition*, p.121.

[51] Allan Stam, *Win, Lose or Draw*, (The University of Michigan Press, Ann Arbor 1996).

[52] Thomas Schelling, *Arms and Influence*, (Yale University Press, New Haven 1966) p.7.

[53] Filson and Werner, 'A Bargaining Model of War and Peace'; Slantchev, 'The Principle of Convergence in Wartime Negotiations'; Powell, 'Bargaining and Learning while Fighting.'

more complete or decisions become more frequent that process converges to a true compromise offer that is beneficial to both sides. By contrast, we have shown that if both players are allowed to make offers and accept them at the time of their choosing they do *not* make any meaningful compromise, at least until the imminent victory of one side. Instead, they both demand that the other accept the minimum it can guarantee itself by continuing to fight, which means complete surrender for much of the war. Most importantly, this divergence persists as uncertainty vanishes and regardless of the identity of the expected winner.

Our rivals are out to identify the weakest of the two sides rather than the weakest among the opponent's types. If one were to make a concession, then all types on the other side would accept sooner. But if the side to offer compromise happens to be stronger than the other, it forgoes the opportunity to see its opponent just take a minimal offer. With a compromise offer on the table, war might be shorter and therefore less costly. But these expected savings do not compensate for the benefit of having a weak type on the other side decide to make do with his best alternative to negotiation. In equilibrium, our rivals implement passive screening for much of the war, offering no concession and hoping that the other side will give in before they do. Only when it is facing imminent defeat does one side begin making significant offers in order to cut short its losses, while adamantly refusing to capitulate. According to our model, wars erupt and endure because the rivals do not rationally seek compromise until the impending victory of one side. This is in stark contrast with the idea that wars are fought to inform on the terms of the bargain that each is willing to accept.

The attrition behavior that our model predicts can be tested for empirically. Indeed, the failure of both parties to offer concessions in equilibrium implies, technically, that the likelihood that each side will give in increases as the flow costs of war incurred *by the other side* increase (see formula (A4) in Appendix).

Langlois and Langlois[54] test this relationship. Specifically, the attrition behavior that our model anticipates involves weighing the expected costs of war against the chances that the other side will give in to one's demands. The lower the *believed* likelihood that the opponent will give in, the lower the flow costs one should be willing to bear for the same prize. If beliefs are an accurate prediction of rational behavior, this implies that higher war flow costs suffered by one side are associated to a higher probability of surrender or concession by the other. This is not a statement about battle deaths or military capabilities as they affect the course of the war. Rather it is a statement about a state's willingness to commit resources to the war effort given the other side's perceived willingness to fight on. To test for attrition behavior we used the proportion of a state's population engaged in military activities, a comprehensive measure of a state's investment in war, as a predictor of the other side's willingness to give in. Our analysis of 72 interstate wars that have occurred between 1823 and 1990 reveals that, once traditional drivers of duration such as battle deaths, capabilities and terrain are accounted for, the proportion of a state's population in the military positively, and significantly, affects the other side's willingness to give in. We therefore conclude that the attrition behavior predicted by our model can helps to explain the duration of interstate wars.

Conclusion

Convergence in war and bargaining models describes a screening process, in which fighting informs, and concessions are calibrated to prompt the separation of types. The convergence of beliefs results in a convergence of offers. To investigate the impact of the discrete-time modeling assumptions made by these models, we have constructed a war and bargaining model that allows the rivals to

[54] Langlois and Langlois, 'Does Attrition Behavior Help Explain the Duration of Interstate Wars?'

bargain freely while fighting in the time continuum. Uncertainty and bargaining are bilateral and the rivals can make offers and counteroffers at the time of their choosing. In our model warfare is an ongoing process that allows for stalemate as well as victory by one side or the other. In this context we find that, in equilibrium, the rivals will adopt uncompromising bargaining positions, preferring to fight it out in the hope that the other side will give in first.

Uncompromising bargaining behavior prevails as long as there is a chance that the other side will surrender under the continued pressure of battle costs. Only when defeat is imminent will the losing side offer small one-sided concessions. In our model, only the dynamic progression of battle towards one side's ultimate demise prompts the loser to sue for peace and offer concessions. As long as defeat is remote or uncertain, the potential for the opponent's surrender drives the rivals to adopt uncompromising positions despite the increasing likelihood that their opponent is resilient. The progress of battle informs on the resilience of the rivals but it does not bring them to the bargaining table. In equilibrium, the convergence of beliefs does not lead to a convergence of settlement offers. The active screening that is at the core of the process of convergence does not take place, and the Principle of Convergence does not hold.

Mathematical Appendix

If i believes that, at time t, all types $j \in [0, \phi_j(t))$ have been screened out then only those types in the segment $[\phi_j(t), 1]$ remain active. If we further write

$$1-\phi_j(t+s) = \big(1-\phi_j(t)\big)\big(1-\phi_j(s|t)\big) \tag{A1}$$

then $\phi_j(s|t)$ represents i's beliefs, *conditional* on reaching time t, about what segment of $[\phi_j(t), 1]$ will have been screened out after a further delay s. Using more generally the notation '$s|t$' for all variables evaluated at $(t+s)$, conditional on reaching time t, one can rewrite i's expected payoff defined in (4) for waiting time $(t+s)$, with $s \geq 0$, as

$\mathbb{E}\pi_i(t+s) = \int_0^t \pi_i(1-y_i(\tau),\tau)d\phi_j(\tau)$

$$+\big(1-\phi_j(t)\big)\big(\sigma_i(t)+e^{-r_it}\mathbb{E}\pi_i(s|t)\big) \tag{A2}$$

with $\quad \mathbb{E}\pi_i(s|t) = \int_0^s \pi_i(1-y_i(\tau|t),\tau|t)d\phi_j(\tau|t)$

$$+(1-\phi_j(s|t))\pi_i(y_j(s|t),s|t) \tag{A3}$$

where $\pi_i(y_j(s|t),s|t)$ is as in (1) but involves only the developments from time t on, and $\sigma_i(t) = -\int_0^t r_ie^{-r_i\tau}c_i(\tau)d\tau$ is the total cost 'sunk' in the war effort by time t.

$\mathbb{E}\pi_i(s|t)$ is i's expected payoff from time t on and it has the very same structure as $\mathbb{E}\pi_i(s)$. Sequential rationality thus requires the maximizing of $\mathbb{E}\pi_i(s|t)$ at each time t over s and offer function $y_i(.|t)$ given beliefs. This is clearly equivalent to optimization from the very start by (A2) as long as $\phi_j(t) < 1$.

For simplicity of exposition we state and prove all claims with initial $t = 0$. In particular, one has $\phi_j(0|t) = 0$ just like $\phi_j(0) = 0$, but this assumes that current belief $\phi_j(t)$ at time t is accounted for by (A2). The general case only requires the burdensome notation '$s|t$' in all statements and proofs below.

A belief-strategy profile is called consistent if beliefs are consistent with strategies according to Bayes' Law. Since offer functions y_i and y_j are *assumed* right-differentiable and piecewise C^1, we find that *consistent* beliefs inherit the same properties. If ϕ_i is continuous and increasing at time t it means that $i = \phi_i(t)$ is the single type i who accepts optimally j's offer $y_j(t)$ at time t. A discontinuity $\Delta\phi_i(t) = \phi_i(t) - \phi_i(t^-) > 0$ could instead mean that a *segment* of types $[\phi_i(t^-), \phi_i(t)]$ accepts $y_j(t)$ at time t. But $\Delta\phi_i(t) > 0$ could also result from signaling, if side $\mathcal{I}$ makes an offer y_i at time t that no type in $[\phi_i(t^-), \phi_i(t))$ could make.[55] However, we will find that all active types i have the same optimal offer y_i and that such signaling is suboptimal. So we focus on acceptance as the updating instrument.

We say that offers 'match' at time t if $y_i(t) + y_j(t) = 1$ but $y_i(s) + y_j(s) < 1$

[55] If only types $i \geq i^*$ can make offer y_i optimally and $\phi_i(t^-) < i^*$ was side $\mathcal{J}$'s belief just before that offer, belief jumps to $\phi_i(t) = i^*$ at time t.

for all $s < t$. When offers match the two sides agree on the split and the game ends.[56] Offers are assumed defined for all times even if they match before the game ends.

For simplicity of exposition, we also assume that $u_i(x) = x$ although all proofs are easily extendable to the case where u_i is a strictly increasing C^1 function.[57] We denote by $\delta(s) = 1 - y_i(s) - y_j(s)$ the spread between the two side's offers. Also for simplicity we denote by $\lambda_{i=\phi_i}(s)$ the quantity $\lambda_i(s)|_{i=\phi_i(s)}$, and similarly for $\xi_{i=\phi_i}$ and $c_{i=\phi_i}$.

We first characterize optimal acceptance times and the resulting consistent beliefs.

Definition 1. *A (belief-strategy) profile is separating for side $\mathcal{I}$ in an interval $\Omega = [0,t)$ if side $\mathcal{J}$'s consistent belief ϕ_i is continuously increasing within that interval.*

Lemma 1. *If the profile is separating for side $\mathcal{I}$ in $\Omega = [0,t)$ then, for all $s \in \Omega$ where offers are differentiable, consistent beliefs must satisfy:*

$$\frac{\phi_j'(s)}{1-\phi_j(s)} = \lambda_{i=\phi_i}(s) \tag{A4}$$

with $\lambda_i(s) = \frac{r_i(y_j(s)+c_i(s))-y_j'(s)}{\delta(s)}$

Proof. Since ϕ_j is a distribution function it is differentiable almost everywhere (a.e.). And since offers are piecewise C^1, i accepts optimally at time $s \in (0,t)$ only if the first order condition

$$\frac{d\mathbb{E}\pi_i(s)}{ds} = \big(\pi_i(1 - y_i(s), s) - \pi_i(y_j(s), s)\big)\phi_j'(s)$$
$$+(1-\phi_j(s))\frac{d}{ds}\pi_i(y_j(s), s) = 0 \tag{A5}$$

holds (a.e). Simplifying (A5) yields

$$\frac{\phi_j'(s)}{1-\phi_j(s)} = \frac{-d\pi_i(y_j(s),s)/ds}{\pi_i(1-y_i(s),s)-\pi_i(y_j(s),s)} = \lambda_i(s) \tag{A6}$$

By consistency of beliefs, (A6) must hold at s for the single type $i = \phi_i(s)$. Since $\lambda_i(s)$ is piecewise continuous, belief ϕ_j is piecewise C^1 (i.e. it inherits

[56] An 'overmatching' offer $y_i(t) > 1 - y_j(t)$ would be quickly accepted by side $\mathcal{J}$ and would be a net loss for any i who could accept $1 - y_j(t)$ instead.

[57] One replaces y_j by $u_i(y_j)$, $1 - y_j$ by $u_i(1 - y_j)$ and y_j' by $u_i'(y_j)y_j'$ in all formulae.

the properties of offers). And (A4) also hold at $s = 0$ in the sense of the right-hand-side derivative. $\square$

By (A4), the inequality $\lambda_{i=\phi_i} \geq 0$ is necessary for ϕ_j to represent side $\mathcal{I}$'s beliefs. Clearly, the sign of λ_i is determined by y_j. But the condition $\lambda_{i=\phi_i} \equiv 0$ within $\Omega = [0, t)$ can occur and implies *constant* beliefs $\phi_j(s) \equiv 0$. However, the condition $\lambda_{i=\phi_i} < 0$ obtains if y_j increases sufficiently fast within Ω. In that case, (A4) cannot hold but we will show that consistent beliefs must then remain constant. Consistent beliefs will also remain constant when none of the types can rationally accept within the given interval. This occurs if offer $y_j(s)$ falls *below* the best alternative $\xi_i(s)$ to negotiation for all active types $i \geq \phi_i(s)$ in Ω. We therefore make:

Definition 2. *Given consistent beliefs, an offer y_i is called in $\Omega = [0, t)$*

(1) 'acceptable' if it satisfies in all Ω

$$y_i(s) \geq \xi_{j=\phi_j}(s) \text{ and } \lambda_{j=\phi_j}(s) \geq 0 \qquad (A7)$$

(2) 'convergent' if both inequalities (A7) are strict *in all Ω*

(3) 'boundary' if (A7) hold with at least one equality *in all Ω;*

(4) 'unacceptable' if at least one of the inequalities (A7) fails in all Ω;

A profile is called 'acceptable' if both *sides' offers are acceptable.*

Lemma 2. *If y_i is unacceptable in $\Omega = [0, t)$ then side $\mathcal{I}$' consistent belief ϕ_j must be constant in Ω. Moreover, unless offer y_j is such that $\lambda_{i=\phi_i} \equiv 0$, side $\mathcal{J}$'s consistent belief ϕ_i must also be constant in Ω.*

Proof. If $y_i(s) < \xi_{j=\phi_j}(s)$ then $j = \phi_j(s)$ cannot optimally accept at s. And if $\lambda_{j=\phi_j}(s) < 0$ then $\frac{d\mathbb{E}\pi_j(s)}{ds} > 0$ (in (A5) written for j) and j cannot optimally accept at s.[58] In either case, no active j can accept and consistent beliefs $\phi_j(s)$ can only be constant in Ω. If y_j is also unacceptable then $\phi_i(s) \equiv 0$ by a symmetric argument. And if y_j is acceptable but such that $\lambda_{i=\phi_i}(s) > 0$ then (with $\mathbb{E}\pi_i(s) = \pi_i(s)$ since $\phi_j(s) \equiv 0$)

[58]Intuitively, offer y_j increases so fast that all types i prefer to wait until it levels off.

$\frac{d\mathbb{E}\pi_i(s)}{ds} = -e^{-r_i s}(r_i\,(y_j(s)+c_i(s)) - y_j'(s)) < 0$ (A8)

at s such that $i = \phi_i(s)$. So, this i cannot accept optimally at that time s and consistent belief ϕ_i can only be constant in Ω.[59] □

To complete our description of consistent beliefs we consider discontinuities in offers.

Lemma 3. *Assume that $\Delta y_i(t) = y_i(t) - y_i(t^-) > 0$. Then side $\mathcal{I}$' consistent belief ϕ_j must be constant in some $\Omega = [t', t)$. Moreover, unless offer y_j is such that $\lambda_{i=\phi_i} \equiv 0$ in Ω, side $\mathcal{J}$'s consistent belief ϕ_i must also be constant in Ω.*

Proof. Let $\Delta\phi_i(t) = \phi_i(t) - \phi_i(t^-) \geq 0$ and let $j = \phi_j(s)$ be the type that accepts optimally at $s \in \Omega$. This requires that $\mathbb{E}\pi_j(s) \geq \mathbb{E}\pi_j(t)$. But

$$\lim_{s\to t^-}(\mathbb{E}\pi_j(t) - \mathbb{E}\pi_j(s)) = \lim_{s\to t^-}\big(\int_s^t \pi_j(1-y_j(s), s)d\phi_i(s) \qquad \text{(A9)}$$
$$+(1-\phi_i(t))\pi_j(y_i(t), t) - (1-\phi_i(s))\pi_j(y_i(s), s)\big)$$
$$= e^{-r_j t}\big(\delta(t^-)\Delta\phi_i(t) + (1-\phi_i(t^-))\Delta y_i(t)\big) > 0$$

So, if $t' < t$ is sufficiently close to t, all $\mathbb{E}\pi_j(s) < \mathbb{E}\pi_j(t)$ and no j can accept optimally in Ω. The same arguments as in Lemma 2 then apply.[60] □

So, a positive jump in offer y_i at t creates an interval *before* t where y_i is also 'unacceptable' with the very same consequences on consistent beliefs. We therefore can extend the definition of unacceptable offer to the case of a discontinuous jump at t, within an adequate interval $\Omega = [t', t)$. A negative drop $\Delta y_i(t) < 0$ can be characterized similarly.[61] In fact, it requires an interval *after* t where the offer is unacceptable in order for beliefs to be consistent. But

[59] The case $\lambda_{i=\phi_i}(s) \equiv 0$ allows an increasing belief ϕ_i while offer y_i is unacceptable. The consequences for belief ϕ_i are only given for the completeness of Lemma 4. But only the consequences on belief ϕ_j are necessary in the Main Theorem at the end.

[60] Intuitively, a jump in y_i is an 'extremely fast' increase which was found unacceptable in Lemma 2.

[61] Consistent beliefs would have to jump according to: $\frac{\Delta\phi_j(t)}{1-\phi_j(t^-)} = 1 - e^{\frac{\Delta y_j(t)}{\delta(t^-)}} > 0$. But the jump $\Delta\phi_j(t) > 0$ would be incompatible with an acceptable profile starting at t. Proofs of these statements are available from the authors upon request.

this last result will not be needed in our proofs since we will only consider drops from a *minimum* acceptable offer that are trivially unacceptable.

We now have a clearer picture of consistent beliefs: unacceptable offers yield constant beliefs within a corresponding interval. And those with a discontinuous jump have a similar effect. Since we will obtain an equilibrium in continuous offers we initially focus on the continuous case and will eventually show that unacceptable, including discontinuous, offers are suboptimal.

Lemma 4. *Given offers y_i and y_j continuous and not matching in the interval $[0,t)$ there exists a unique pair of consistent beliefs $\phi_i = \mathcal{M}_i[y_i, y_j]$ and $\phi_j = \mathcal{M}_j[y_i, y_j]$. Moreover, beliefs depend continuously on offers.*[62]

Proof. We can rewrite (A4) in view of Lemmata 2 and 3

$$\phi_j' = f_j(\phi_i, \phi_j, s) = (1-\phi_j)\rho(s)\lambda_{i=\phi_i}(s) \tag{A10}$$

$$\text{with } \rho(s) = \begin{cases} 0 & \text{if } \lambda_{i=\phi_i}(s) < 0 \text{ or } y_i \text{ or } y_j \text{ unacceptable (but } \lambda_{j=\phi_j}(s) > 0) \\ 1 & \text{otherwise} \end{cases}$$

and symmetrically for $\phi_i' = f_i(\phi_i, \phi_j, s)$. Clearly, f_i and f_j are piecewise continuous in s. And they have bounded partial derivatives in ϕ_i and ϕ_j. Given initial $\phi_i(0) = \phi_j(0) = 0$ this first order system of differential equations has a unique solution by Picard's Theorem.[63] The solution satisfies

$$\phi_j(s) = 1 - e^{-\int_0^s \rho(\tau)\lambda_{i=\phi_i}(\tau)d\tau} \tag{A11}$$

This clearly defines a distribution function and therefore beliefs. Moreover ϕ_j in (A11) fits the conditions of Lemmata 1, 2 and 3, and therefore defines the unique consistent beliefs. Finally, $\lambda_{i=\phi_i}$ and $\lambda_{j=\phi_j}$ depend continuously on offers and so do ϕ_i and ϕ_j.[64] □

[62]The operators $\mathcal{M}_i$ and $\mathcal{M}_j$ are continuous in the topology of uniform convergence (or any weaker topology).

[63]See Earl Coddington and Norman Levinson, *Theory of Ordinary Differential Equations*, (McGraw-Hill, New York, 1955) p.12. Bounded partial derivatives guarantee the standard Lipschwitz condition.

[64]We can define $\phi_j(t) = \lim_{s\to t^-} \phi_j(s)$. If offers match at time t, then ϕ_j' becomes infinite and $\phi_j(t) = 1$ provided $\lambda_{i=\phi_i} > 0$. If beliefs need to be defined for $s \geq t$ we simply set them

We now restrict our attention to *continuous acceptable* profiles and obtain necessary conditions of optimality within that set. We still denote victory time (for either side) by $\theta \leq \infty$. The case $\theta = \infty$ means battlefield stalemate.

Lemma 5. *If side $\mathcal{J}$ expects to win the war at time θ, or if $\theta = \infty$, then the minimum acceptable offer y_j^* for side $\mathcal{J}$ is $y_j^*(t) \equiv 0$.*

Proof. Since side $\mathcal{I}$ cannot win $\xi_i(t) \equiv 0$ for all i. And since y_j^* yields $\lambda_i(t) > 0$ for all i and t, it is acceptable. Since no offer can be negative this is the minimum. □

Lemma 6. *If side $\mathcal{I}$ expects to lose the war at time $\theta < \infty$ then, given any continuous offer y_j, there exists a unique minimum (continuous acceptable) offer y_i^* for side $\mathcal{I}$.*

Proof. For any continuous y_i, let $\phi_j = \mathcal{M}_j[y_i, y_j]$ be the unique consistent beliefs obtained in Lemma 4. Now let $\eta_i = \mathcal{N}_i[\phi_j]$ be defined for all $s \in [0, \theta]$ by:

$$\eta_i(s) = max\{0, e^{r_j(s-\theta)} - \int_s^\theta r_j e^{r_j(s-\tau)} c_{j=\phi_j}(\tau) d\tau\} \quad \text{(A12)}$$

If we let

$$\mathcal{S}_i = \{y_i \in \mathcal{C}^1([0, \theta] \to [0, 1]) | \forall s \in [0, \theta] : 0 \leq y_i'(s) \leq r_j(1 + c_{j=0}(s))\}$$

then $\eta_i \in \bar{\mathcal{S}}_i$ (the topological closure of $\mathcal{S}_i$). $\mathcal{S}_i$, having uniformly bounded variations, is equicontinuous. And since $[0, \theta]$ is compact, by Ascoli's Theorem[65] $\bar{\mathcal{S}}_i$ is compact. And it is clearly convex. Moreover, both $\mathcal{M}_j$ and $\mathcal{N}_i$ are continuous operators and so is the composite

$$\mathcal{O}_i = \mathcal{N}_i \circ \mathcal{M}_j : y_i \to \phi_j = \mathcal{M}_j[y_i, y_j] \to \eta_i = \mathcal{N}_i[\phi_j]$$

So, the operator $\mathcal{O}_i$ from $\bar{\mathcal{S}}_i$ into $\bar{\mathcal{S}}_i$ is compact and convex. By Schauder's Theorem[66] it has a fixed point $y_i^* = \mathcal{O}_i[y_i^*] \in \bar{\mathcal{S}}_i$. Wherever positive, y_i^* must therefore satisfy, with $\phi_j^* = \mathcal{M}_j[y_i^*, y_j]$ as consistent belief

$$y_i^*(s) = e^{r_j(s-\theta)} - \int_s^\theta r_j e^{r_j(s-\tau)} c_{j=\phi_j^*}(\tau) d\tau \quad \text{(A13)}$$

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as the constant $\phi_j(s) \equiv \phi_j(t)$.

[65] Halsey Royden, *Real Analysis*, (Macmillan, New York, 1968) p.157.

[66] Klaus Deimling, *Nonlinear Functional Analysis*, (Springer-Verlag, Berlin, 1985) p. 60.

which is the solution of $\lambda_{j=\phi_j}(s) \equiv 0$ together with $y_i^*(\theta) = 1$. Moreover,

$$y_i^*(s) \geq e^{r_j(s-\theta)} - \int_s^\theta r_j e^{r_j(s-\tau)} c_j(\tau)|_{j=\phi_j^*(s)} d\tau = \xi_{j=\phi_j}(s) \tag{A14}$$

since $c_{j=\phi_j^*}(\tau) = c_j(\tau)|_{j=\phi_j^*(\tau)} \leq c_j(\tau)|_{j=\phi_j^*(s)}$ for all $\tau \geq s$. So y_i^* is indeed acceptable.

We now show that y_i^* is the *minimum* continuous acceptable offer in $[0, \theta]$.

Whenever $y_i^*(s) = 0$ this is trivial. Otherwise, for any $y_i(s) > 0$ the necessary conditions $\lambda_{j=\phi_j} \geq 0$ and $y_i(\theta) = 1$ for y_i to be acceptable yield, by integration, for all $s \in [0, \theta]$:

$$y_i(s) \geq e^{r_j(s-\theta)} - \int_s^\theta r_j e^{r_j(s-\tau)} c_{j=\phi_j}(\tau) d\tau \tag{A15}$$

If, in addition, $y_i(s) \leq y_i^*(s)$ for all such s, then one easily verifies that

$\phi_j = \mathcal{M}_j[y_i, y_j] \leq \mathcal{M}_j[y_i^*, y_j] = \phi_j^*$

Since this yields $c_{j=\phi_j}(s) \geq c_{j=\phi_j^*}(s)$ we must have by (A13) and (A15) $y_i(s) \geq y_i^*(s)$. So, no acceptable $y_i(s)$ can be *strictly* less than $y_i^*(s)$. Since this holds for all $s \in [0, \theta]$, y_j^* must be unique.[67] □

To obtain the No-Concession Result central to all our findings we need:

Lemma 7. *For any (continuous acceptable) profile in an interval $\Omega = [0, t)$ and for all active types $i \in \mathcal{I}$ and for all $s \in \Omega$:*

$$\mathbb{E}\pi_i(s) = y_j(0) + \epsilon r_i \int_0^s e^{-r_i\tau} c_{\mathcal{I}}(\tau)(1 - \phi_j(\tau)) (i - \phi_i(\tau)) d\tau \tag{A16}$$

Proof. Since $\frac{d\mathbb{E}\pi_i(s)}{ds}$ in (A5) must be piecewise continuous, we may write

$$\mathbb{E}\pi_i(s) = \mathbb{E}\pi_i(0) + \int_0^s \frac{d\mathbb{E}\pi_i(\tau)}{d\tau} d\tau \tag{A17}$$

Replacing (A4) into (A17) and simplifying yields (A16).[68] □

Lemma 8. *(No Concession): If y_i is a best-reply offer (for all active types i) to y_j in a continuous acceptable profile then it must be the minimum y_i^*.*

[67] It is entirely possible that y_i^* and y_j match at some $t < \theta$. In that case, belief ϕ_i may (or may not) reach the value 1 in the interval $[t, \theta]$. Also note that y_j need not be acceptable in this argument.

[68] If the game has ended by time $t \geq \theta$, we have $c_{\mathcal{I}}(\tau|t) \equiv 0$, the integrand in (A7) is nil and $\mathbb{E}\pi_i(s|t) = y_j(t) \equiv y_j(\theta)$ where $y_j(\theta)$ is the outcome of war or the agreed upon bargain at time θ.

Proof. First assume that offer y_i is convergent in some interval denoted $\Omega = [0, t)$ and let $t_i \geq 0$ be such that type $i = \phi_i(t_i)$ accepts optimally at time $t_i \in \Omega$. Also denote by $L_i[y_i]$ the integrand in (A16) evaluated at $s = t_i$. Type i seeks an offer y_i that maximizes $L_i[y_i]$.[69] Now, for all $s \in \Omega$, let $\eta(s) = s - t$, and

$$y_i^h(s) = \begin{cases} y_i(s) + h\eta(s) & \text{if } s \in \Omega \\ y_i(s) & \text{otherwise} \end{cases} \tag{A18}$$

By continuity, for $h > 0$ small enough, $y_i^h \geq \xi_{j=\phi_j}$, $\lambda_{j=\phi_j} \geq 0$ and $\lambda_{i=\phi_i} \geq 0$ in Ω.[70] So, $y_i^h < y_i$ is a continuous acceptable offer in Ω. Furthermore

$$\frac{d}{dh} L_i[y_i^h] = -\int_0^{t_i} e^{-r_i s} c_{\mathcal{I}}(s) \big((1 - \phi_j(s)) \tfrac{\partial}{\partial h}\phi_i(s) + (i - \phi_i(s)) \tfrac{\partial}{\partial h}\phi_j(s)\big) ds$$
$$+ e^{-r_i t_i} c_{\mathcal{I}}(t_i)(1 - \phi_j(t_i))(i - \phi_i(t_i)) \tfrac{\partial}{\partial h} t_i \tag{A19}$$

Since the last term is nil by $i = \phi_i(t_i)$,[71] we only need to show that the integrand is negative at $h = 0$ so that y_i^h is strictly better than y_i for $h > 0$ small enough. We have

$$\tfrac{\partial}{\partial h}\phi_i(s) = (1 - \phi_i(s)) \int_0^s \tfrac{\partial}{\partial h}\lambda_{j=\phi_j}(\tau) d\tau \tag{A20}$$

and $\quad \tfrac{\partial}{\partial h}\phi_j(s) = (1 - \phi_j(s)) \int_0^s \tfrac{\partial}{\partial h}\lambda_{i=\phi_i}(\tau) d\tau$

Moreover, omitting all time arguments for simplicity

$$\tfrac{\partial}{\partial h}\lambda_{j=\phi_j} = \frac{\partial \lambda_j}{\partial y_i}\frac{dy_i^h}{dh} + \frac{\partial \lambda_j}{\partial y_i'}\frac{dy_i^{h\prime}}{dh} + \frac{\partial \lambda_j}{\partial j}\frac{d\phi_j}{dh} \tag{A21}$$
$$= \tfrac{1}{\delta}\Big((r_j + \lambda_{j=\phi_j})\eta - \eta' - \epsilon c_{\mathcal{J}} \tfrac{\partial}{\partial h}\phi_j\Big)$$

and $\quad \tfrac{\partial}{\partial h}\lambda_{i=\phi_i} = \frac{\partial \lambda_i}{\partial y_i}\frac{dy_i^h}{dh} + \frac{\partial \lambda_i}{\partial i}\frac{d\phi_i}{dh} = \tfrac{1}{\delta}\Big(\eta\lambda_{i=\phi_i} - \epsilon c_{\mathcal{I}} \tfrac{\partial}{\partial h}\phi_i\Big) \qquad$ (A22)

Clearly, both above quantities are bounded in Ω and we can let[72]

$$\gamma = \sup_{\Omega}\{\big|\tfrac{\partial}{\partial h}\lambda_{j=\phi_j}\big|, \big|\tfrac{\partial}{\partial h}\lambda_{i=\phi_i}\big|\}$$

so that $\sup_{\Omega}\{\big|\tfrac{\partial}{\partial h}\phi_i\big|, \big|\tfrac{\partial}{\partial h}\phi_j\big|\} \leq \gamma t$.

Replacing in (A21) and (A22) yields, since $\eta \leq 0$,

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[69] This is a typical 'Optimal Control' problem. The standard technique for solving such problems is Variational Calculus and the Pontryagin Maximum Principle. Unfortunately, this is unsuccessful here because of the non-linearity of (A4). Indeed, the resulting 'costate equations' are not solvable. However, we do use a method borrowed from Variational Calculus in (A19).

[70] If $\lambda_{i=\phi_i} = 0$ then y_i^h has no effect on $\lambda_{i=\phi_i}$ which remains nil.

[71] $\frac{\partial t_i}{\partial h}$ can obtained by differentiating the identity $i = \phi_i(t_i)$.

[72] We must choose t so that offers are not matching or γ could become unbounded.

$\frac{\partial}{\partial h}\lambda_{j=\phi_j} \leq \frac{1}{\delta}(-1+\epsilon c_{\mathcal{J}}\gamma t)$ and $\frac{\partial}{\partial h}\lambda_{i=\phi_i} \leq \frac{1}{\delta}\epsilon c_{\mathcal{I}}\gamma t$ (A23)

It follows that, with $0 < \delta^* \leq \delta(s)$, $0 < c_{\mathcal{I}}(s) \leq c^*_{\mathcal{I}}$ and $0 < c_{\mathcal{J}}(s) \leq c^*_{\mathcal{J}}$ on Ω

$$(1-\phi_j(s))\frac{\partial}{\partial h}\phi_i(s) + (i-\phi_i(s))\frac{\partial}{\partial h}\phi_j(s) \quad \text{(A24)}$$

$$= (1-\phi_j(s))\Big((1-\phi_i(s))\int_0^s \frac{\partial}{\partial h}\lambda_{j=\phi_j}d\tau + (i-\phi_i(s))\int_0^s \frac{\partial}{\partial h}\lambda_{i=\phi_i}d\tau\Big)$$

$$\leq \frac{s}{\delta^*}(1-\phi_j(s))\Big((1-\phi_i(s))(-1+\epsilon c^*_{\mathcal{J}}\gamma t) + (i-\phi_i(s))\epsilon c^*_{\mathcal{I}}\gamma t\Big) < 0$$

for $t > 0$ small enough (with all quantities evaluated at $h = 0$).

So, $\frac{d}{dh}L_i[y_i^h]|_{h=0} > 0$ and $y_i^h < y_i$ for $h > 0$ small enough is a *strictly* better continuous acceptable offer for all types i *who accept* in Ω. Moreover, if a convergent y_i is best for some active types who *do not accept* in Ω, let $i^* \geq \phi_i(t)$ be the infimum of all such types. If any type $i \geq i^*$, makes a convergent offer y_i in Ω it 'signals out' all types $i < i^*$ and consistent belief ϕ_i must be updated so that we may now let $i^* = \phi_i(0) = 0$. Since belief ϕ_j is unaffected, the offer y_i is still convergent in Ω. But, by the above argument, y_i can't be optimal.[73]

Second, assume instead that y_i is a boundary offer other than y_i^* in some interval Ω. If the current time (denoted 0) is close enough to θ this requires $y_i \geq y_i^* \geq \xi_{j=\phi_j} > 0$, with resulting $\lambda_{j=\phi_j} \equiv 0$ and therefore $\phi_i \equiv 0$ in Ω. So, y_i is strictly increasing. Without loss of generality we can assume that $y_i(t) = 1$ at time $t < \theta$, otherwise it would be equal to y_i^*. Solving $\lambda_{j=\phi_j} = 0$ with $y_i(t) = 1$ yields, for all $s \in \Omega$

$$y_i(s) = e^{r_j(s-t)} - \int_s^t r_j e^{r_j(s-\tau)} c_{j=\phi_j}(\tau)d\tau \quad \text{(A25)}$$

with consistent beliefs $\phi_j = \mathcal{M}_j[y_i, y_j]$ and $\phi_i \equiv 0$ in Ω. It follows that

$$\frac{\partial y_i(s)}{\partial t} = -r_j e^{r_j(s-t)}(1+c_{j=\phi_j}(t)) - \int_s^t r_j e^{r_j(s-\tau)}\frac{\partial c_j}{\partial j}\frac{\partial \phi_j}{\partial t}d\tau < 0 \quad \text{(A26)}$$

as long as $\frac{\partial \phi_j}{\partial t} \leq 0$ since $\frac{\partial c_j}{\partial j} < 0$. But

$$\frac{\partial}{\partial t}\phi_j(s) = (1-\phi_j(s))\int_0^s \frac{\partial \lambda_i}{\partial y_i}\frac{\partial y_i}{\partial t}d\tau < 0 \quad \text{(A27)}$$

since $\frac{\partial \lambda_i}{\partial y_i} > 0$. It follows that an increase in t results in a uniform decrease in y_i and a corresponding uniform decrease in ϕ_j while ϕ_i remains constant. Therefore

[73] Indeed, signaling by making any offer y_i other than the minimum acceptable one turns out to be suboptimal.

$$\frac{d}{dt}L_i[y_i] = -\int_0^{t_i} e^{-r_i s} c_{\mathcal{I}}(s)(i - \phi_i(s))\frac{\partial}{\partial t}\phi_j(s)d\tau > 0 \quad \text{(A28)}$$

and a decreased boundary y_i is strictly better for all active types i.[74]

The only offer that cannot be improved upon and is therefore the only possible best reply for all active types is the minimum y_i^*. □

We have only obtained *necessary* conditions for best-reply offers within the set of continuous acceptable profiles. In the next result we show that there exists an equilibrium within that set and characterize it. We then show that it is a MPBE of the game without the restriction to continuous acceptable offers.

Lemma 9. *Within the set of continuous acceptable profiles there exists a unique MPBE and it is made up of the minimum acceptable offers y_i^* and y_j^*.*

Proof. If there is no victory time $\theta < \infty$ we pick an arbitrary (large but finite) θ and impose the restriction that $y_i(s) \equiv y_i(\theta)$ for all $s \geq \theta$. Since decreasing acceptable offers are convergent and cannot be best replies by Lemma 8, we can limit our search for an optimum to the set $\mathcal{A}_i$ of non-decreasing continuous acceptable offers. Since $\mathcal{A}_i$ is a closed subset of $\bar{\mathcal{S}}_i$ (in Lemma 6) it is compact. Since the operator $L_i[y_i]$ is continuous in y_i it must reach a maximum over $\mathcal{A}_i$. If side $\mathcal{I}$ is not losing, this maximizand can only be the (unique) minimum acceptable $y_i^* \equiv 0$ that is common to all active types i. If side $\mathcal{I}$ is losing then side $\mathcal{J}$ is winning and $y_j^* \equiv 0$. But there is a unique maximizand y_i^* of $L_i[y_i]$ in response to y_j^* by Lemma 6 for all active types i. And if there is a stalemate $y_i^* \equiv 0$ is the optimum for any finite θ and therefore for all times. □

We denote by $\mathcal{E}$ the pair (y_i^*, y_j^*) formed by the above minima together with consistent beliefs (ϕ_i^*, ϕ_j^*).[75] We now show that $\mathcal{E}$ is in fact an *unrestricted* MPBE.

Theorem. *The profile $\mathcal{E}$ of Lemma 9 forms a MPBE of the game.*

[74] Optimal $t_i \leq t$ here is the time at which offers match and can only increase with t.

[75] Of course, all arguments are valid in the '$s|t$' sense and therefore hold for any current history and beliefs, thus ensuring sequential rationality.

Proof. First observe that a discontinuous jump in offer is unacceptable in some interval Ω by Lemma 3. And a discontinuous drop from the *minimum* acceptable offer can only be unacceptable. So, we only need to show that unacceptable offers are suboptimal. Consider an interval $[0, t)$ where y_i is unacceptable in response to y_j^*. By Lemma 2, consistent belief ϕ_j become constant (equal to 0) in Ω and the expected payoff for any active type i reduces to $\mathbb{E}\pi_i(s) = \pi_i(y_j^*(s))$ in Ω. If we denote by $\mathbb{E}\pi_i^*(s)$ the expected payoff under $\mathcal{E}$ we have two cases for y_j^*:

(1) $\lambda_{i=\phi_i^*} \equiv 0$ in Ω. In that case $\phi_j^* \equiv 0$ and $\mathbb{E}\pi_i^*(s) \equiv \pi_i(y_j^*(s)) = \mathbb{E}\pi_i(s)$. So, there can be no improvement for any type i in switching to y_i;

(2) $y_j^* \equiv 0$ and all $\lambda_i > 0$. In that case, one easily verifies that for all $s \in \Omega$:

$(1 - \phi_j)(\lambda_{i=\phi_i^*} - \lambda_i) > -\lambda_i$

But this is equivalent to $\mathbb{E}\pi_i^*(s) > \pi_i(y_j^*(s)) = \mathbb{E}\pi_i(s)$. So, all active types i's expected payoffs are uniformly decreased in all Ω by the switch to y_i. Moreover, all conditional expected payoffs $\mathbb{E}\pi_i(s|t)$ in (A3) are unchanged since y_i is unchanged after time t. So, all expected payoffs are uniformly decreased *for all times* under the switch. In particular, *optimal* expected payoffs are uniformly decreased. And this holds for any t. □