Appendix One: Nash's Theorem*

A (finite) game in normal form is defined by a list \( \{1, \ldots, i, \ldots, n\} \) of \( n \) players, a set of pure strategies \( \mathcal{A}_i = \{\alpha^i_1, \ldots, \alpha^i_{k(i)}\} \) for each player, and a payoff function \( \pi_i : \mathcal{A} = \bigoplus_{i=1}^n \mathcal{A}_i \rightarrow \mathbb{R} \) for each player which associates the payoff \( \pi_i(\alpha^1, \ldots, \alpha^n) \) for player \( i \), to any pure strategy \( n \)-tuple \( \{\alpha^1, \ldots, \alpha^n\} \).

A mixed strategy \( x^i \) for player \( i \) is a probability distribution \( \{x^i_1, \ldots, x^i_{k(i)}\} \) over \( i \)'s pure strategies. A mixed strategy \( n \)-tuple \( x = \{x^1, \ldots, x^n\} \) is a list of \( n \) probability distributions (one for each player over that player's pure strategy set). The set \( B \) of all \( n \)-tuples \( x \) is clearly a compact (closed and bounded) and convex subset of the Euclidean space \( \mathbb{R}^n \) with \( m = \sum_{i=1}^n k(i) \).

The expected payoff resulting from a mixed strategy \( n \)-tuple \( x \) is the multilinear extension of \( \pi_i \) (i.e., it is linear in each \( x^i \)). It will be convenient to denote by \( \pi_i(x^i, x^{-i}) \) the expected payoff when player \( i \) uses \( x^i \) and all other players together use the \((n-1)\)-tuple \( x^{-i} \).

We will prove:

**Nash's Theorem:** Any (finite) game in normal form has a Nash equilibrium.

We will need:

**Lemma:** \( x \in B \) is a Nash equilibrium if and only if for any \( i \) and any \( \alpha \in \mathcal{A}_i \)

\[ \pi_i(\alpha, x^{-i}) \leq \pi_i(x^i, x^{-i}) \]  \hspace{1cm} (1)

**Proof:** Clearly, no pure strategy \( \alpha \) is a better reply than \( x^i \) to \( x^{-i} \) for player \( i \) if (1) holds. Thus, for any mixed strategy \( y^i = \{y^i_\alpha | \alpha \in \mathcal{A}_i\} \) and by linearity for each \( i \):

\[ \pi_i(y^i, x^{-i}) = \sum_{\alpha \in \mathcal{A}_i} y^i_\alpha \pi_i(\alpha, x^{-i}) \leq \left( \sum_{\alpha \in \mathcal{A}_i} y^i_\alpha \right) \pi_i(x^i, x^{-i}) \leq \pi_i(x^i, x^{-i}) \]  \hspace{1cm} (2)

and \( y^i \) is no better than \( x^i \) in response to \( x^{-i} \). \( Q.E.D. \)

**Proof of the theorem:** For any pure strategy \( \alpha \), let \( i(\alpha) \) be such that \( \alpha \in \mathcal{A}_i \), and let:

\[ u_\alpha(x) = \max \{0, \pi_i(\alpha, x^{-i}) - \pi_i(x^i, x^{-i})\} \]  \hspace{1cm} (3)

for \( i = i(\alpha) \). Clearly, \( u_\alpha \) is continuous in \( x \) since \( \pi_i \) is. Further let \( \mathcal{A}(\alpha) = \mathcal{A}_i \) for \( i = i(\alpha) \) and:

\[ y_\alpha = \frac{x_\alpha + u_\alpha}{1 + \sum_{\beta \in \mathcal{A}(\alpha)} u_\beta} \]  \hspace{1cm} (4)

where \( x_\alpha \) is the probability of \( \alpha \) in \( x \). Evidently, \( y_\alpha \geq 0 \) and \( \sum_{\alpha \in \mathcal{A}(\alpha)} y_\alpha = 1 \). We can thus define \( y^i \) as a probability distribution of components \( y_\alpha \) for \( \alpha \in \mathcal{A}(\alpha) \), and let \( y = \{y^1, \ldots, y^n\} \) be the corresponding \( n \)-tuple of mixed strategies. Moreover, since \( y_\alpha \) is...

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clearly continuous in \( x \), so is \( y \) as a function of \( x \). Therefore, by Brouwer’s theorem, the continuous map \( \phi : B \to B \) defined by \( y = \phi(x) \) must have a fixed point \( z = \phi(z) \in B \).

We finally verify that \( z \) is a Nash equilibrium. We first observe that for any \( i \) there must exist at least one \( \alpha \in A_i \) such that \( z_\alpha > 0 \) and \( u_\alpha = 0 \). Indeed, were it not the case, we would have \( \pi_i(\alpha, z^{-i}) > \pi_i(z^i, z^{-i}) \) for all \( z_\alpha > 0 \) and thus:

\[
\pi_i(z^i, z^{-i}) = \sum_{\alpha \in A_i} z_\alpha \pi_i(\alpha, z^{-i}) > \left( \sum_{\alpha \in A_i} z_\alpha \right) \pi_i(z^i, z^{-i}) = \pi_i(z^i, z^{-i})
\]

(5)

a contradiction. But if \( z_\alpha > 0 \) and \( u_\alpha = 0 \) for some \( \alpha \in A_i \) then, for that \( \alpha \):

\[
z_\alpha = \frac{z_\alpha}{1 + \sum_{\beta \in A(\alpha)} u_\beta}
\]

(6)

so that \( \sum_{\beta \in A(\alpha)} u_\beta = 0 \) and \( u_\beta = 0 \) for all \( \beta \in A(\alpha) \). It follows that

\[
\pi_i(\beta, z^{-i}) \leq \pi_i(z^i, z^{-i})
\]

for all \( \beta \in A_i \) and for all \( i \). By the above lemma, \( z \) is a Nash equilibrium. Q.E.D.