Math 370 Midterm #2 Sample

1. Show that any cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ (with $a \neq 0, b, c, d$ real) has at least one real root.
Hint: Assuming $a > 0$, rewrite $p(x) = x^3.r(x)$ and argue that there exists $M > 0$ such that $r(x) \geq \frac{1}{2}a$ for $|x| \geq M$.

2. Assume that the sequences $\{x_n\}$ and $\{y_n\}$ both converge to the same limit L.
Define the sequence $\{z_n\}$ by:

$$z_n = \begin{cases} x_k & \text{if } n = 2k-1 \\ y_k & \text{if } n = 2k \end{cases}$$

Using the definition of convergent sequences, prove that $\{z_n\}$ converges to L.

3. Assume that $f : \mathcal{I} = [a,b] \to R$ is continuous and that there exists a unique $c \in \mathcal{I}$ such that $f(x) = 0 \Leftrightarrow x = c$. Assume that $\{x_n\}$ is a sequence in $\mathcal{I}$ such that $\lim_{n\to\infty} f(x_n) = 0$. Prove that $\{x_n\}$ converges to c using the following theorem: "if all convergent subsequences of a bounded sequence converge to a same limit c then the sequence converges to c."[1]

4. Assume that $g : R \to R$ satisfies $\forall x, y :\ g(x+y) = g(x).g(y)$. Show that if g is continuous at 0 then g is continuous everywhere in R.

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[1]This theorem appears in other textbooks but not yours. The proof is by contradiction: if $\{x_n\}$ does not converge to c it has (by Bolzano-Weierstrass) a subsequence that converges but not to c.