Math 370 Midterm #2 Sample

1. Show that any cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ (with $a \neq 0, b, c, d$ real) has at least one real root. Hint: Assuming $a > 0$, rewrite $p(x) = x^3.r(x)$ and argue that there exists $M > 0$ such that $r(x) \geq \frac{1}{2}a$ for $|x| \geq M$.
$r(x) = a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}$ clearly approaches a as $|x| \to \infty$ since $\frac{1}{x}$ approaches 0. So, there must exist $M > 0$ such that $r(x) \geq \frac{1}{2}a$ for $|x| \geq M$.
Now, $p(M) = M^3 r(M) \geq \frac{1}{2}aM^3 > 0$ and $p(-M) = -M^3 r(M) \leq -\frac{1}{2}aM^3 < 0$.
Since the continuous function p changes sign on the interval $[-M, M]$ it must have a root by the intermediate value theorem.

2. Assume that the sequences $\{x_n\}$ and $\{y_n\}$ both converge to the same limit L.
Define the sequence $\{z_n\}$ by:

$$z_n = \begin{cases} x_k & \text{if } n = 2k-1 \\ y_k & \text{if } n = 2k \end{cases}$$

Using the definition of convergent sequences, prove that $\{z_n\}$ converges to L.
$\forall \epsilon > 0\ \exists \mathrm{K}_x \in N$ such that $k \geq \mathrm{K}_x \Rightarrow |x_k - L| < \epsilon$
and (for the same ϵ) $\exists \mathrm{K}_y \in N$ such that $k \geq \mathrm{K}_y \Rightarrow |y_k - L| < \epsilon$
So, if I let $\mathrm{N} = \max\{2\mathrm{K}_x - 1, 2\mathrm{K}_y\}$ then $n \geq \mathrm{N} \Rightarrow \{|z_n - L| = |x_k - L| < \epsilon$ if $n = 2k-1$ or $|z_n - L| = |y_k - L| < \epsilon$ if $n = 2k\}$.

3. Assume that $f : \mathcal{I} = [a, b] \to R$ is continuous and that there exists a unique $c \in \mathcal{I}$ such that $f(x) = 0 \Leftrightarrow x = c$. Assume that $\{x_n\}$ is a sequence in $\mathcal{I}$ such that $\lim_{n\to\infty} f(x_n) = 0$. Prove that $\{x_n\}$ converges to c using the following theorem: "if all convergent subsequences of a bounded sequence converge to a same limit c then the sequence converges to c."[1]
Since x_n is bounded consider an arbitrary convergent subsequence $\{x_{n_k}\}$ with limit x.
Since $\{f(x_{n_k})\}$ is a subsequence of the convergent sequence $\{f(x_n)\}$ it has the same limit 0. By the sequential characterization theorem $f(x) = \lim_{k\to\infty} f(x_{n_k}) = 0$. By assumption it follows that $x = c$. By the theorem, since all convergent subsequences $\{x_{n_k}\}$ have the same limit c, $\{x_n\}$ converges to c.

4. Assume that $g : R \to R$ satisfies $\forall x, y :\ g(x+y) = g(x).g(y)$. Show that if g is continuous at 0 then g is continuous everywhere in R.
If g is continous at 0 then (by definition) $\lim_{x\to 0} g(x) = g(0)$. But then:

$$\lim_{z\to y} g(z) = \lim_{x\to 0} g(x+y) \qquad \text{(since } z \to y \Leftrightarrow z = x + y \text{ and } x \to 0\text{)}$$
$$= \lim_{x\to 0} \big(g(x).g(y)\big) = \big(\lim_{x\to 0} g(x)\big).g(y) \qquad \text{(by limit theorems)}$$
$$= g(0).g(y) = g(0+y) = g(y)$$

So, g is continuous at any y.

---

[1] This theorem appears in other textbooks but not yours. The proof is by contradiction: if $\{x_n\}$ does not converge to c it has (by Bolzano-Weierstrass) a subsequence that converges but not to c.